ASYMPTOTIC REPRESENTATION OF RATIO STATISTICS AND THEIR MEAN SQUARED ERRORS

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Some statistics in common use take a form of a ratio of two statistics such as sample correlation coefficient, Pearson’s coefficient of variation and so on. In this paper, obtaining an asymptotic representation of the ratio statistic until the third order term, we will discuss asymptotic mean squared errors of the ratio statistics. We will also discuss bias correction of the sample correlation coefficient and the sample coefficient of variation. Mean squared errors of the corrected estimators are also obtained.

Key words and phrases: Asymptotic mean squared error, asymptotic $U$-statistics, bias correction, coefficient of variation, correlation coefficient, $H$-decomposition.

1. Introduction

Let $X_1, \ldots, X_n$ be independently and identically distributed random vectors with distribution function $F$. Let $T_n = T_n(X_1, \ldots, X_n)$ and $S_n = S_n(X_1, \ldots, X_n)$ be statistics related to parameters $t_n$ and $s_n$. Some statistics in common use take a form of a ratio of two statistics, $T_n / S_n$, such as the sample correlation coefficient, the Pearson’s coefficient of variation, odds ratio, etc. In this paper we will obtain an asymptotic representation of the ratio statistic with remainder term $o_p(n^{-3/2})$ which satisfies

$$\limsup_{n \to \infty} P\{|n^{3/2}o_p(n^{-3/2})| \geq \varepsilon\} = 0$$

for any $\varepsilon > 0$. Using this asymptotic representation, an asymptotic mean squared error with remainder term $o(n^{-2})$ is established and we will discuss asymptotic mean squared errors of the sample correlation coefficient and the Pearson’s coefficient of variation.

Let us assume that

$$T_n = t_n + n^{-1} \delta_T + n^{-2} \sum_{i=1}^{n} \tau_0(X_i) + n^{-1} \sum_{i=1}^{n} \tau_1(X_i)$$

$$+ n^{-2} \sum_{i \neq j} \tau_2(X_i, X_j) + n^{-3} \sum_{i \neq j \neq k} \tau_3(X_i, X_j, X_k) + o_p(n^{-3/2})$$

and

$$S_n = s_n + n^{-1} \delta_S + n^{-2} \sum_{i=1}^{n} \zeta_0(X_i) + n^{-1} \sum_{i=1}^{n} \zeta_1(X_i)$$


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\[ + \sum_{C_{n,2}} n^{-2} \zeta_2(X_i, X_j) + n^{-3} \sum_{C_{n,3}} \zeta_3(X_i, X_j, X_k) + o_p(n^{-3/2}) \]

where

\begin{align}
E[\tau_0(X_1)] &= E[\zeta_0(X_1)] = E[\tau_1(X_1)] = E[\zeta_1(X_1)] = 0, \\
E[\tau_2(X_1, X_2) | X_1] &= E[\zeta_2(X_1, X_2) | X_1] = 0 \quad \text{a.s.}, \\
E[\tau_3(X_1, X_2, X_3) | X_1, X_2] &= E[\zeta_3(X_1, X_2, X_3) | X_1, X_2] = 0 \quad \text{a.s.}
\end{align}

and \( \delta_T \) and \( \delta_S \) are constant biases. \( \sum_{C_{n,k}} \) indicates that the summation is taken over all integers \( i_1, \ldots, i_k \) satisfying \( 1 \leq i_1 < i_2 < \cdots < i_r \leq n \). Many statistics satisfy these assumptions, and Lai and Wang (1993) called them asymptotic \( U \)-statistics. Since some parameters depend on \( n \), such as the variance and the central third moment of \( U \)-statistic etc., we assume that

\begin{align}
t_n &= t^{(0)} + n^{-1}t^{(1)} + O(n^{-2}) \\
s_n &= s^{(0)} + n^{-1}s^{(1)} + O(n^{-2}) \quad (s^{(0)} \neq 0)
\end{align}

A typical example of the ratio statistic \( T_n/S_n \) is the sample correlation coefficient \( r_n \) which is constituted from a covariance estimator and variance estimators. Using computer program package, Knott and Frangos (1983) calculated an asymptotic variance \( n \text{Var}(r_n) \) when the underlying distribution \( F \) is bivariate normal. Here applying the asymptotic representation to the sample correlation coefficient, we will obtain general form of the asymptotic representation and asymptotic mean squared error without assuming the normality. We also discuss the bias correction of the sample coefficient and its asymptotic mean squared error. Another example of the ratio statistic is the sample coefficient of variation. We will obtain asymptotic representation and mean squared error of the coefficient, and discuss the bias correction.

In Section 2, we will obtain the asymptotic representation of \( T_n/S_n \) with remainder term \( o_p(n^{-3/2}) \) and we will discuss the asymptotic mean squared error of \( T_n/S_n \). In Sections 3 and 4, we will consider the applications to the sample correlation coefficient and Pearson’s coefficient of variation, and also discuss bias corrections of them.

### 2. Asymptotic representation and mean squared error

Using \( H \)-decomposition and the moment evaluation of them, we will obtain the asymptotic representation of \( T_n/S_n \). Let us assume the following moment conditions

\begin{align}
E[|\tau_0(X_1)|^3 + |\zeta_0(X_1)|^3] < \infty, \\
E[|\tau_1(X_1)|^{4+\varepsilon} + |\tau_2(X_1, X_2)|^{4+\varepsilon} + |\tau_3(X_1, X_2, X_3)|^{4+\varepsilon}] < \infty \\
\text{and} \\
E[|\zeta_1(X_1)|^{4+\varepsilon} + |\zeta_2(X_1, X_2)|^{4+\varepsilon} + |\zeta_3(X_1, X_2, X_3)|^{4+\varepsilon}] < \infty
\end{align}

(2.1) \( E[|\tau_0(X_1)|^3 + |\zeta_0(X_1)|^3] < \infty \),

(2.2) \( E[|\tau_1(X_1)|^{4+\varepsilon} + |\tau_2(X_1, X_2)|^{4+\varepsilon} + |\tau_3(X_1, X_2, X_3)|^{4+\varepsilon}] < \infty \)

and

(2.3) \( E[|\zeta_1(X_1)|^{4+\varepsilon} + |\zeta_2(X_1, X_2)|^{4+\varepsilon} + |\zeta_3(X_1, X_2, X_3)|^{4+\varepsilon}] < \infty \)
Then we have the following representation.

Let us define

$$
\delta = \frac{\delta_T}{s(0)} - \frac{t(0)\delta_S}{(s(0))^2} - \frac{E[\tau_1(X_1)\zeta_1(X_1) - \tau_1(X_1)\zeta_1(X_1) + t(0)\zeta^2(X_1)]}{(s(0))^3},
$$

$$
\eta_0(x) = \frac{\tau_0(x)}{s(0)} - \frac{t(0)\zeta_0(x)}{(s(0))^2} + \left\{ \frac{E[\zeta^2(X_1)]}{(s(0))^3} - \frac{\delta_S + s(1)}{(s(0))^2} \right\} \tau_1(x)
+ \left\{ \frac{2t(0)s(1)}{(s(0))^3} + \frac{2t(0)\delta_S + 2E[\tau_1(X_1)\zeta_1(X_1)]}{(s(0))^3} \right\} \zeta_1(x)
- \frac{1}{(s(0))^2} \{ E[\tau_1(X_2)\zeta_2(x, X_2) + \zeta_1(X_2)\tau_2(x, X_2)]
+ \tau_1(x)\zeta_1(x) - E[\tau_1(X_1)\zeta_1(X_1)] \}
+ \frac{t(0)}{(s(0))^3} \{ \zeta^2(X_1) - E[\zeta^2(X_1)] + 2E[\zeta_1(X_2)\zeta_2(x, X_2)] \},
$$

$$
\eta_1(x) = \frac{\tau_1(x)}{s(0)} - \frac{t(0)\zeta_1(x)}{(s(0))^2},
$$

$$
\eta_2(x, y) = \frac{\tau_2(x, y)}{s(0)} - \frac{1}{(s(0))^2} \{ \tau_1(x)\zeta_1(y) + \tau_1(y)\zeta_1(x) + t(0)\zeta_2(x, y) \}
+ \frac{2t(0)\zeta_1(x)\zeta_1(y)}{(s(0))^3},
$$

$$
\eta_3(x, y, z) = \frac{\tau_3(x, y, z)}{s(0)}
- \frac{1}{(s(0))^2} \{ t(0)\zeta_3(x, y, z) + \zeta_1(x)\tau_2(y, z) + \zeta_1(y)\tau_2(x, z)
+ \zeta_1(z)\tau_2(x, y) + \tau_1(x)\zeta_2(y, z)
+ \tau_1(y)\zeta_2(x, z) + \tau_1(z)\zeta_2(x, y) \}
+ \frac{1}{(s(0))^3} \{ 2t(0)\zeta_1(x)\zeta_2(y, z) + \zeta_1(y)\zeta_2(x, z) + \zeta_1(z)\zeta_2(x, y) \}
+ 2\big[ \tau_1(x)\zeta_1(y)\zeta_1(z) + \tau_1(y)\zeta_1(x)\zeta_1(z) 
+ \tau_1(z)\zeta_1(x)\zeta_1(y) \big]
+ \frac{6t(0)\zeta_1(x)\zeta_1(y)\zeta_1(z)}{(s(0))^4}
$$

and

$$
U_n = \frac{t_n}{s_n} + n^{-1}\delta + n^{-2} \sum_{i=1}^{n} \eta_0(X_i) + n^{-1} \sum_{i=1}^{n} \eta_1(X_i)
+ n^{-2} \sum_{C_{n, 2}} \eta_2(X_i, X_j) + n^{-3} \sum_{C_{n, 3}} \eta_3(X_i, X_j, X_k).
$$

Then we have the following representation.
Theorem 1. Assume that the conditions (1.1)∼(1.7) and (2.1)∼(2.3) are satisfied. Then we have
\[ \frac{T_n}{S_n} = U_n + o_p(n^{-3/2}). \]

Proof. See Appendix.

\( U_n \) is an approximation of the ratio statistic \( T_n/S_n \) until the third order term and using the form \( U_n \), we can study the asymptotic properties of the statistic \( T_n/S_n \). Here we will consider the asymptotic mean squared error of \( U_n \). It follows from the conditions (1.3), (1.4) and (1.5) that
\[ E[\eta_0(X_1)] = E[\eta_1(X_1)] = 0, \quad E[\eta_2(X_1, X_2) \mid X_1] = 0 \quad \text{a.s.} \]
and
\[ E[\eta_3(X_1, X_2, X_3) \mid X_1, X_2] = 0 \quad \text{a.s.} \]

Thus we can obtain the asymptotic mean squared error \( AMSE(T_n/S_n) \) as follows.

Theorem 2. Under the same assumptions of Theorem 1, we have
\[ AMSE \left( \frac{T_n}{S_n} \right) = n^{-1}E[\eta_1^2(X_1)] + n^{-2} \left\{ \delta^2 + 2E[\eta_0(X_1)\eta_1(X_1)] + \frac{1}{2}E[\eta_2^2(X_1, X_2)] \right\} + O(n^{-3}). \]

Proof. From Theorem 1, we have
\[ AMSE \left( \frac{T_n}{S_n} \right) = E \left( U_n - \frac{t_n}{s_n} \right)^2 = E(n^{-1}\delta + n^{-2}A_0 + n^{-1}A_1 + n^{-2}A_2 + n^{-3}A_3)^2 \]
where
\[ A_0 = \sum_{i=1}^{n} \eta_0(X_i), \quad A_1 = \sum_{i=1}^{n} \eta_1(X_i), \quad A_2 = \sum_{C_{n,2}} \eta_2(X_i, X_j) \]
and
\[ A_3 = \sum_{C_{n,3}} \eta_3(X_i, X_j, X_k). \]

From the equations (2.4) and (2.5), we have
\[ E(A_0A_2) = E(A_0A_3) = E(A_1A_2) = E(A_1A_3) = E(A_2A_3) = 0, \]
\[ E(n^{-2}A_0)^2 = O(n^{-3}), \quad E(n^{-3}2A_0A_1) = n^{-2}2E[\eta_0(X_1)\eta_1(X_1)], \]
\[ E(n^{-1}A_1)^2 = n^{-1}E[\eta_1^2(X_1)], \quad E(n^{-2}A_2)^2 = \frac{(n-1)}{2n^3}E[\eta_2^2(X_1, X_2)] \]
and \( E(n^{-3}A_3)^2 = O(n^{-3}) \). Thus we have the equation (2.6).

Let us define

\[ e_1 = E[\tau_1^2(X_1)], \quad e_2 = E[\zeta_1^2(X_1)], \quad e_3 = E[\tau_0(X_1)\tau_1(X_1)], \]
\[ e_4 = E[\tau_0(X_1)\zeta_1(X_1)], \quad e_5 = E[\tau_1(X_1)\zeta_0(X_1)], \quad e_6 = E[\zeta_0(X_1)\zeta_1(X_1)], \]
\[ e_7 = E[\tau_1(X_1)\zeta_1(X_1)], \quad e_8 = E[\zeta_1^3(X_1)], \quad e_9 = E[\tau_1(X_1)\zeta_1^2(X_1)], \]
\[ e_{10} = E[\tau_1^2(X_1)\zeta_1(X_1)], \quad e_{11} = E[\zeta_1(X_1)\zeta_1(X_2)\zeta_2(X_1, X_2)], \]
\[ e_{12} = E[\tau_1(X_1)\zeta_1(X_2)\zeta_2(X_1, X_2)], \quad e_{13} = E[\tau_1(X_1)\tau_1(X_2)\zeta_2(X_1, X_2)], \]
\[ e_{14} = E[\zeta_1(X_1)\zeta_1(X_2)\zeta_2(X_1, X_2)], \quad e_{15} = E[\tau_1(X_1)\zeta_1(X_2)\tau_2(X_1, X_2)], \]
\[ e_{16} = E[\tau_2^2(X_1, X_2)], \quad e_{17} = E[\zeta_2^2(X_1, X_2)] \]

and
\[ e_{18} = E[\tau_2(X_1, X_2)\zeta_2(X_1, X_2)]. \]

Then from the direct computations, we can obtain an explicit form as follows.

\[
(2.7) \quad E[\eta_1^2(X_1)] = \frac{1}{(s^{(0)})^2} e_1 - \frac{2t^{(0)}}{(s^{(0)})^3} e_7 + \frac{(t^{(0)})^2}{(s^{(0)})^4} e_2
\]

and
\[
(2.8) \quad \delta^2 + 2E[\eta_0(X_1)\eta_1(X_1)] + \frac{1}{2} E[\eta_2^2(X_1, X_2)]
\]
\[
= \frac{1}{2(s^{(0)})^2} (2\delta_T^2 + 4e_3 + e_{16})
\]
\[
- \frac{1}{(s^{(0)})^3} \{2(s^{(1)} + \delta_S)e_1 + 2(t^{(1)} + 2\delta_T)e_7 + 2(e_{10} + e_{13} + 2e_{15})
\]
\[
+ t^{(0)}(2\delta_T\delta_S + 2e_4 + 2e_5 + e_{18})\}
\]
\[
+ \frac{1}{2(s^{(0)})^4} \{6(e_1e_2 + 2e_7^2)
\]
\[
+ 4t^{(0)}[t^{(1)} + 2\delta_T]e_2 + (3s^{(1)} + 4\delta_S)e_7 + 2e_9 + 4e_{12} + 2e_{14}
\]
\[
+ (t^{(0)})^2(2\delta_S^2 + 4e_6 + e_{17})\}
\]
\[
- \frac{2}{(s^{(0)})^5} \{9t^{(0)}e_2e_7 + (t^{(0)})^2[(2s^{(1)} + 3\delta_S)e_2 + e_8 + 3e_{11}]\}
\]
\[
+ \frac{9(t^{(0)})^2e_2^2}{(s^{(0)})^6}.
\]

In the following sections, we will discuss applications of the above results.

3. **Correlation coefficient**

Let \( \{X_i\}_{i \geq 1} \) be two dimensional random vectors. Putting \( X_i = (Y_i, Z_i) \), we denote

\[
\text{Var}(X_1) = \text{Var} \left\{ \begin{pmatrix} Y_1 \\ Z_1 \end{pmatrix} \right\} = \begin{pmatrix} \sigma_y^2 & \rho \sigma_y \sigma_z \\ \rho \sigma_y \sigma_z & \sigma_z^2 \end{pmatrix}.
\]
Let us consider the estimation of the correlation coefficient $\rho$. Define

$$ T_n = (n - 1)^{-1} \sum_{i=1}^{n} (Y_i - \bar{Y})(Z_i - \bar{Z}) $$

and

$$ S_n = \left\{ (n - 1)^{-2} \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \sum_{i=1}^{n} (Z_i - \bar{Z})^2 \right\}^{1/2} $$

where $\bar{Y} = \sum Y_i / n$ and $\bar{Z} = \sum Z_i / n$. Then $r_n = T_n / S_n$ is a sample correlation coefficient. Assuming that the underlying distribution is bivariate normal and using a computer program package, Knott and Frangos (1983) calculated an asymptotic variance $n \text{Var}(r_n)$. Here applying the Theorems 1 and 2, we will obtain an asymptotic representation of $r_n$, without assuming the normality. Let us define

$$ \mu_{k\ell} = E[(Y_1 - E(Y_1))^k (Z_1 - E(Z_1))^\ell], \quad k, \ell = 0, 1, \ldots, 6, $$

$$ \tilde{y}_i = y_i - E(Y_i) \quad \text{and} \quad \tilde{z}_i = z_i - E(Z_i). $$

Further let us define

$$ \tau_1(x_1) = \tilde{y}_1 \tilde{z}_1 - \rho \sigma_y \sigma_z, \quad \tau_2(x_1, x_2) = -(\tilde{y}_1 \tilde{z}_2 + \tilde{y}_2 \tilde{z}_1), $$

$$ \delta_S = -\frac{\sigma_z \mu_{40}}{8 \sigma_y^3} - \frac{\sigma_y \mu_{04}}{8 \sigma_z^3} + \frac{\mu_{22}}{4 \sigma_y \sigma_z}, $$

$$ \zeta_0(x_1) = -\frac{1}{8 \sigma_y \sigma_z} \left\{ \frac{3 \sigma_4^4 \mu_{40} - \sigma_y^4 \mu_{04} - 2 \sigma_y^2 \sigma_z^2 \mu_{22}}{16 \sigma_y^5 \sigma_z^3} \tilde{y}_1^2 + \frac{3 \sigma_4^4 \mu_{04} - \sigma_y^4 \mu_{40} - 2 \sigma_y^2 \sigma_z^2 \mu_{22}}{16 \sigma_y^3 \sigma_z^5} \tilde{z}_1^2 + \frac{(\sigma_y^4 \mu_{30} - \sigma_y^2 \sigma_z^2 \mu_{12}) \tilde{y}_1 + (\sigma_y^4 \mu_{03} - \sigma_y^2 \sigma_z^2 \mu_{21}) \tilde{z}_1}{2 \sigma_y^3 \sigma_z^3} \right\}, $$

$$ \zeta_1(x_1) = \frac{\sigma_z^2 \tilde{y}_1^2 + \sigma_y^2 \tilde{z}_1^2 - 2 \sigma_y^2 \sigma_z^2}{2 \sigma_y \sigma_z}, $$

$$ \zeta_2(x_1, x_2) = -\frac{(\sigma_y^2 \tilde{y}_1^2 - \sigma_y^2 \tilde{z}_1^2)(\sigma_y^2 \tilde{y}_2^2 - \sigma_y^2 \tilde{z}_2^2)}{4 \sigma_y^3 \sigma_z^3} - \frac{\sigma_z \tilde{y}_1 \tilde{y}_2}{\sigma_y} - \frac{\sigma_y \tilde{z}_1 \tilde{z}_2}{\sigma_z} $$

and

$$ \zeta_3(x_1, x_2, x_3) = \frac{1}{8 \sigma_y^3 \sigma_z^3} \left\{ (\sigma_y^2 \tilde{y}_1^2 + \sigma_y^2 \tilde{z}_1^2 - 2 \sigma_y^2 \sigma_z^2)(\sigma_y^2 \tilde{y}_2^2 - \sigma_y^2 \tilde{z}_2^2)(\sigma_y^2 \tilde{y}_3^2 - \sigma_y^2 \tilde{z}_3^2) \right\} $$

$$ + (\sigma_y^2 \tilde{y}_1^2 + \sigma_y^2 \tilde{z}_1^2 - 2 \sigma_y^2 \sigma_z^2)(\sigma_y^2 \tilde{y}_2^2 - \sigma_y^2 \tilde{z}_2^2)(\sigma_y^2 \tilde{y}_3^2 - \sigma_y^2 \tilde{z}_3^2) $$

$$ + (\sigma_y^2 \tilde{y}_2^2 + \sigma_y^2 \tilde{z}_2^2 - 2 \sigma_y^2 \sigma_z^2)(\sigma_y^2 \tilde{y}_1^2 - \sigma_y^2 \tilde{z}_1^2)(\sigma_y^2 \tilde{y}_3^2 - \sigma_y^2 \tilde{z}_3^2) $$

$$ + (\sigma_y^2 \tilde{y}_3^2 + \sigma_y^2 \tilde{z}_3^2 - 2 \sigma_y^2 \sigma_z^2)(\sigma_y^2 \tilde{y}_1^2 - \sigma_y^2 \tilde{z}_1^2)(\sigma_y^2 \tilde{y}_2^2 - \sigma_y^2 \tilde{z}_2^2) \right\}. $$
Thus from Theorem 1, we have the asymptotic representation of the correlation

\[ n \]

and from Theorem 2, we can obtain the asymptotic mean squared error of \( n \). The \( n^{-1} \) term of the asymptotic mean squared error is given by

\[
\delta^2 + 2E[\eta_0(X_1)\eta_1(X_1)] + \frac{1}{2} E[\eta_2(X_1, X_2)]
\]

For the \( n^{-2} \) term, it follows from the equation (2.8) and direct computation that

\[
= 1 + \frac{\mu_{40}\mu_{22} + 2\mu_{31}^2}{\sigma_y^2\sigma_z^2} + \frac{\mu_{22}\mu_{40} + 2\mu_{73}^2}{\sigma_y^6\sigma_z^2} + \frac{\mu_{22}^2 + 2\mu_{31}\mu_{13}}{\sigma_y^4\sigma_z^4}
\]

\[
- \frac{\mu_{42} - 2\mu_{30}\mu_{12} - 4\mu_{21}^2}{\sigma_y^4\sigma_z^2} - \frac{\mu_{24} - 2\mu_{21}\mu_{03} - 4\mu_{12}^2}{\sigma_y^2\sigma_z^4} - \frac{\mu_{22}}{\sigma_y^2\sigma_z^2}
\]

\[
+ \rho \left\{ \frac{33\mu_{40}\mu_{31}}{8\sigma_y\sigma_z} - \frac{33\mu_{13}\mu_{04}}{8\sigma_y^5\sigma_z^5} - \frac{13\mu_{40}\mu_{13} + 26\mu_{31}\mu_{22}}{8\sigma_y^7\sigma_z^7} - \frac{13\mu_{31}\mu_{04} + 26\mu_{22}\mu_{13}}{8\sigma_y^5\sigma_z^5} - \frac{5\mu_{51} - 30\mu_{30}\mu_{21}}{4\sigma_y^5\sigma_z} + \frac{5\mu_{15} - 30\mu_{12}\mu_{03}}{4\sigma_y\sigma_z^5}
\]

\[
+ \frac{3\mu_{33} - 15\mu_{21}\mu_{12} - 3\mu_{30}\mu_{03}}{2\sigma_y^3\sigma_z^3} + \frac{\mu_{31}}{\sigma_y^3\sigma_z} + \frac{\mu_{13}}{4\sigma_y\sigma_z^2} \right\}
\]
Since each term of $\hat{\mu}_{k\ell}$ is a ratio statistic again, using Theorem 1, we can obtain an asymptotic representation of $\hat{\delta}/n$ as

$$n^{-1}\hat{\delta} = n^{-1}\delta + n^{-2}\sum_{i=1}^{n} \varphi(X_i) + o_p(n^{-3/2})$$

where

$$\varphi(x_1) = \rho \left\{ \frac{3(\tilde{y}_1^4 - \mu_{40})}{\sigma_y^4} + \frac{3(\tilde{z}_1^4 - \mu_{04})}{\sigma_z^4} + \frac{2(\tilde{y}_1^2 \tilde{z}_1^2 - \mu_{22})}{\sigma_y^2 \sigma_z^2} \right\} + \frac{3\mu_{40}}{8\sigma_y^4} + \frac{3\mu_{04}}{8\sigma_z^4} + \frac{\mu_{22}}{4\sigma_y^2 \sigma_z^2} \right\} \eta_1(x_1)

+ \frac{3\mu_{31}}{4\sigma_y^3 \sigma_z} + \frac{\mu_{13}}{4\sigma_y^3 \sigma_z^3} - \rho \left( \frac{3\mu_{40}}{4\sigma_y^6} + \frac{\mu_{22}}{4\sigma_y^4 \sigma_z^2} \right) (\tilde{y}_1^2 - \sigma_y^2)

+ \frac{\mu_{31}}{4\sigma_y^3 \sigma_z^3} + \frac{3\mu_{13}}{4\sigma_y^3 \sigma_z^2} - \rho \left( \frac{3\mu_{04}}{4\sigma_y^6} + \frac{\mu_{22}}{4\sigma_y^4 \sigma_z^2} \right) (\tilde{z}_1^2 - \sigma_z^2)

- \frac{\tilde{y}_1^2 \tilde{z}_1 - \mu_{31}}{2\sigma_y^3 \sigma_z} - \frac{\tilde{y}_1 \tilde{z}_1^3 - \mu_{13}}{2\sigma_y \sigma_z^3}.$$
Thus we have
\[ r_n^* = r_n - n^{-1} \delta - n^{-2} \sum_{i=1}^{n} \varphi(X_i) + o_p(n^{-3/2}) \]
and so
\[
AMSE(r_n^*) = AMSE(r_n) - 2E \left[ (r_n - \rho) \left\{ n^{-1} \delta + n^{-2} \sum_{i=1}^{n} \varphi(X_i) \right\} \right] \\
+ E \left[ \left\{ n^{-1} \delta + n^{-2} \sum_{i=1}^{n} \varphi(X_i) \right\}^2 \right] \\
= AMSE(r_n) - n^{-2} \{ \delta^2 + 2E[\varphi(X_1)\eta(X_1)] \} + O(n^{-3}).
\]
Using this equation, we can obtain a mean squared error of \( r_n^* \). From direct computation, we have
\[
E[\varphi(X_1)\eta(X_1)] \\
= \frac{3\mu_{40}\mu_{22} + 6\mu_{31}^2}{8\sigma_y^6\sigma_z^2} + \frac{3\mu_{22}\mu_{04} + 6\mu_{13}^2}{8\sigma_y^2\sigma_z^6} \\
+ \frac{\mu_{22}^2 + 2\mu_{31}\mu_{13}}{4\sigma_y^4\sigma_z^4} - \frac{\mu_{42}}{2\sigma_y^2\sigma_z^2} - \frac{\mu_{24}}{2\sigma_y^2\sigma_z^4} \\
+ \rho \left\{ \frac{3\mu_{40}\mu_{31}}{2\sigma_y^2\sigma_z} - \frac{3\mu_{13}\mu_{04}}{2\sigma_y^4\sigma_z^2} - \frac{\mu_{40}\mu_{13} + 2\mu_{31}\mu_{22}}{2\sigma_y^5\sigma_z^3} \\
- \frac{\mu_{31}\mu_{04} + 2\mu_{22}\mu_{13}}{2\sigma_y^3\sigma_z^2} + \frac{5\mu_{51}}{8\sigma_y^6\sigma_z} + \frac{5\mu_{15}}{8\sigma_y^4\sigma_z^2} + \frac{3\mu_{33}}{4\sigma_y^3\sigma_z^3} \right\} \\
+ \rho^2 \left\{ \frac{15\mu_{40}^2}{32\sigma_y^8} + \frac{15\mu_{04}^2}{32\sigma_y^8} + \frac{3\mu_{40}\mu_{22}}{4\sigma_y^6\sigma_z^2} + \frac{3\mu_{22}\mu_{04}}{4\sigma_y^2\sigma_z^6} \\
+ \frac{3\mu_{40}\mu_{04} + 6\mu_{22}^2}{16\sigma_y^4\sigma_z^4} - \frac{5\mu_{42}}{16\sigma_y^2\sigma_z^2} - \frac{5\mu_{24}}{16\sigma_y^2\sigma_z^4} - \frac{3\mu_{60}}{16\sigma_y^6} - \frac{3\mu_{06}}{16\sigma_y^6} \right\}. 
\]
Using these results, we will discuss the asymptotic mean squared errors of \( r_n \) and \( r_n^* \) when the underlying distribution is the bivariate normal and elliptical distributions.

[Bivariate normal]

Here we consider the case of the bivariate normal distribution
\[
X_i = \begin{pmatrix} Y_i \\ Z_i \end{pmatrix} \sim N \left( \theta, \begin{pmatrix} \sigma_y^2 & \rho \sigma_y \sigma_z \\ \rho \sigma_y \sigma_z & \sigma_z^2 \end{pmatrix} \right).
\]
Since
\[
\mu_{30} = \mu_{03} = \mu_{21} = \mu_{12} = 0, \quad \mu_{40} = 3\sigma_y^4, \quad \mu_{04} = 3\sigma_z^4, \\
\mu_{31} = 3\rho \sigma_y^3 \sigma_z, \quad \mu_{13} = 3\rho \sigma_y \sigma_z^3, \quad \mu_{22} = (1 + 2\rho^2) \sigma_y^2 \sigma_z^2, \\
\mu_{60} = 15\sigma_y^6, \quad \mu_{06} = 15\sigma_z^6, \quad \mu_{51} = 15\rho \sigma_y^5 \sigma_z, \quad \mu_{15} = 15\rho \sigma_y \sigma_z^5, \\
\mu_{42} = 3(1 + 4\rho^2) \sigma_y^4 \sigma_z^2, \quad \mu_{24} = 3(1 + 4\rho^2) \sigma_y^2 \sigma_z^4, \quad \mu_{33} = 3\rho(3 + 2\rho^2) \sigma_y^3 \sigma_z^3,
\]
substituting these values, we have the asymptotic mean squared error

\[
AMSE(r_n) = n^{-1}(1 - \rho^2)^2 + n^{-2}(1 - \rho^2)^2 \left( 1 + \frac{23}{4} \rho^2 \right).
\]

(3.1)

Since the bias \( \delta = \rho(\rho^2 - 1)/2 \), we have the asymptotic variance of \( r_n \)

\[
\text{Var}(r_n) \approx AMSE(r_n) - n^{-2} \delta^2
\]

\[
= n^{-1}(1 - \rho^2)^2 + n^{-2}(1 - \rho^2)^2 \left( 1 + \frac{11}{2} \rho^2 \right).
\]

This coincides with the result of Knott and Frangos (1983, p. 502), who have obtained it backed by the computer program package.

Further we can show that

\[
\delta = \rho(\rho^2 - 1)/2 \quad \text{and} \quad 2E[\varphi(X_1)\eta_1(X_1)] = (1 - \rho^2)^2(3\rho^2 - 1).
\]

Thus from (2.7), we have the asymptotic mean squared error of \( r_n^* \)

\[
AMSE(r_n^*) = AMSE(r_n) - n^{-2}\{\delta^2 + 2E[\varphi(X_1)\eta_1(X_1)]\} + O(n^{-3})
\]

\[
= n^{-1}(1 - \rho^2)^2 + n^{-2}(1 - \rho^2)^2 \left( 2 + \frac{5}{2} \rho^2 \right) + O(n^{-3}).
\]

Since

\[
AMSE(r_n) - AMSE(r_n^*) = \frac{1}{4n^2}(1 - \rho^2)^2(13\rho^2 - 4) + O(n^{-3}),
\]

if the underlying distribution is normal and \(|\rho| \geq 2/\sqrt{13} (= 0.555)\), \( r_n^* \) is superior to \( r_n \) from viewpoint of unbiasedness and asymptotic mean squared error. It is possible to correct the bias by another methods like jackknife correction. But those corrected estimators may coincide with \( r_n^* \) until the remainder term \( o_p(n^{-3/2}) \). Thus the asymptotic mean squared error takes the same one of \( r_n^* \).

[Elliptical distributions]

Let us consider the case of the elliptical distribution \( E_2(\mu, \Lambda) \). The density and characteristic functions are

\[
f(x; \mu, \Lambda) = c|\Lambda|^{-1/2}g\{(x - \mu)'\Lambda^{-1}(x - \mu)\}
\]

for some function \( g \) and constant \( c \), and

\[
\phi(t) = \exp(it'\mu)\psi(t'\Lambda t)
\]

for some function \( \psi \). Note that

\[
E(X_i) = \mu \quad \text{and} \quad \text{Var}(X_i) = \begin{pmatrix} \sigma_y^2 & \rho \sigma_y \sigma_z \\ \rho \sigma_y \sigma_z & \sigma_z^2 \end{pmatrix} = -2\psi'(0)\Lambda.
\]
It follows from Maruyama and Seo (2003) that

\[ \mu_{30} = \mu_{03} = \mu_{21} = \mu_{12} = 0, \]
\[ \mu_{40} = \frac{\psi^{(2)}(0)}{\psi'(0)}^2 3\sigma_y^4, \quad \mu_{04} = \frac{\psi^{(2)}(0)}{\psi'(0)}^2 3\sigma_z^4, \quad \mu_{31} = \frac{\psi^{(2)}(0)}{\psi'(0)}^2 3\rho\sigma_y^3\sigma_z, \]
\[ \mu_{13} = \frac{\psi^{(2)}(0)}{\psi'(0)}^2 3\rho\sigma_y\sigma_z^3, \quad \mu_{22} = \frac{\psi^{(2)}(0)}{\psi'(0)}^2 (1 + 2\rho^2)\sigma_y^2\sigma_z^2, \]
\[ \mu_{60} = \frac{\psi^{(3)}(0)}{\psi'(0)}^3 15\sigma_y^6, \quad \mu_{06} = \frac{\psi^{(3)}(0)}{\psi'(0)}^3 15\sigma_z^6, \quad \mu_{51} = \frac{\psi^{(3)}(0)}{\psi'(0)}^3 15\rho\sigma_y^5\sigma_z, \]
\[ \mu_{15} = \frac{\psi^{(3)}(0)}{\psi'(0)}^3 15\rho\sigma_y\sigma_z^5, \quad \mu_{42} = \frac{\psi^{(3)}(0)}{\psi'(0)}^3 3(1 + 4\rho^2)\sigma_y^4\sigma_z^2, \]
\[ \mu_{24} = \frac{\psi^{(3)}(0)}{\psi'(0)}^3 3(1 + 4\rho^2)\sigma_y^2\sigma_z^4, \quad \mu_{33} = \frac{\psi^{(3)}(0)}{\psi'(0)}^3 3\rho(3 + 2\rho^2)\sigma_y^3\sigma_z^3. \]

Thus we have the asymptotic mean squared error of \( r_n \) as follows

\[ AMSE(r_n) = \frac{n^{-1}}{\psi'(0)^2} (1 - \rho^2)^2 + n^{-2}(1 - \rho^2)^2 \left\{ 1 - \frac{\psi^{(2)}(0)}{\psi'(0)}^2 - 6 \frac{\psi^{(3)}(0)}{\psi'(0)}^2 + \frac{\psi^{(2)}(0)}{\psi'(0)}^4 \left( 7 + \frac{23}{4} \rho^2 \right) \right\}. \]

And the difference of \( AMSE(r_n) \) and \( AMSE(r_n^*) \) is

\[ AMSE(r_n) - AMSE(r_n^*) = \delta^2 + 2E[\varphi(X_1)\eta_1(X_1)] + O(n^{-3}) \]
\[ = \frac{\psi^{(2)}(0)}{\psi'(0)^4} \left( 5 - 3\rho^2 - 3\rho^4 + \frac{13}{4} \rho^6 \right) \]
\[ + \frac{\psi^{(3)}(0)}{\psi'(0)^3} (-6 + 12\rho^2 - 6\rho^4) + O(n^{-3}). \]

If the distribution is the bivariate normal, \( \psi(t) = \exp(-t/2) \) and so

\[ \frac{\psi^{(2)}(0)}{\psi'(0)^2} = 1 \quad \text{and} \quad \frac{\psi^{(3)}(0)}{\psi'(0)^3} = 1. \]

For the bivariate \( t \)-distribution which has the density function

\[ f(x) = \frac{\Gamma((\nu + 2)/2)}{\Gamma(\nu/2)\nu\pi} |\Lambda|^{-1/2} \left\{ 1 + \frac{1}{\nu} (x - \mu)'\Lambda^{-1}(x - \mu) \right\}^{-(\nu+2)/2}, \]

we have (see Maruyama and Seo (2003))

\[ \frac{\psi^{(2)}(0)}{\psi'(0)^2} = \frac{(\nu - 2)^2}{(\nu - 4)(\nu - 2)} \quad \text{and} \quad \frac{\psi^{(3)}(0)}{\psi'(0)^3} = \frac{(\nu - 2)^3}{(\nu - 6)(\nu - 4)(\nu - 2)}. \]
In the case of \( \nu = 8 \), we have
\[
\delta^2 + 2E[\varphi(X_1) \eta_1(X_1)] = \frac{9}{16}(-28 + 84\rho^2 - 60\rho^4 + 13\rho^6)
\]
and if \( \rho > 0.6963 \), \( r_n^* \) is superior to \( r_n \) from viewpoint of unbiasedness and asymptotic mean squared error. In the case of \( \nu = 20 \), we have
\[
\delta^2 + 2E[\varphi(X_1) \eta_1(X_1)] = \frac{9}{224}(-52 + 300\rho^2 - 276\rho^4 + 91\rho^6)
\]
and if \( \rho > 0.4601 \), \( r_n^* \) is superior than \( r_n \).

For the contaminated bivariate normal distribution which has the density function
\[
f(x) = \frac{1 - \omega}{2\pi} |\Lambda|^{-1/2} \exp \left\{ -\frac{1}{2}(x - \mu)' \Lambda^{-1}(x - \mu) \right\}
+ \frac{\omega}{2\pi\gamma^2} |\Lambda|^{-1/2} \exp \left\{ -\frac{1}{2\gamma^2}(x - \mu)' \Lambda^{-1}(x - \mu) \right\}
\]
\((0 \leq \omega \leq 1, 0 < \gamma)\), we have (see Maruyama and Seo (2003))
\[
\frac{\psi^{(2)}(0)}{\{\psi'(0)\}^2} = \frac{1 + \omega(\gamma^4 - 1)}{(1 + \omega(\gamma^2 - 1))^2} \quad \text{and} \quad \frac{\psi^{(3)}(0)}{\{\psi'(0)\}^3} = \frac{1 + \omega(\gamma^6 - 1)}{(1 + \omega(\gamma^2 - 1))^3}.
\]
In the case of \( \gamma = 3, \omega = 0.1 \), we have
\[
\delta^2 + 2E[\varphi(X_1) \eta_1(X_1)] = \frac{25}{324}(-484 + 1668\rho^2 - 1284\rho^4 + 325\rho^6)
\]
and if \( \rho > 0.6362 \), \( r_n^* \) is superior than \( r_n \) from viewpoint of unbiasedness and asymptotic mean squared error. In the case of \( \gamma = 3, \omega = 0.7 \), we have
\[
\delta^2 + 2E[\varphi(X_1) \eta_1(X_1)]
= \frac{25}{701538156}(-58782284 + 453908268\rho^2 - 443175084\rho^4 + 156159575\rho^6)
\]
and if \( \rho > 0.3878 \), \( r_n^* \) is superior than \( r_n \).

4. **Coefficient of variation**

Here we will consider the Pearson’s coefficient of variation. For mean \( \mu = E(X_1) \neq 0 \) and variance \( \sigma^2 = V(X_1) \), the coefficient of variation is given by
\[
\frac{\sigma}{\mu}.
\]
Using the sample mean \( \bar{X} \) and the unbiased sample variance \( \hat{\sigma}^2 = (n - 1)^{-1} \sum (X_i - \bar{X})^2 \), we have the sample coefficient of variation
\[
V_n = \frac{\hat{\sigma}}{\bar{X}}.
\]
Thus we can apply the results of Theorems 1 and 2.

Let us define
\[ \mu_k = E[(X_1 - \mu)^k] \quad (k = 3, \sim, 6) \quad \text{and} \quad \tilde{x}_i = x_i - \mu. \]

Further define
\[ \delta_T = -\frac{\mu_4 - \sigma^4}{8\sigma^3}, \]
\[ \tau_0(x_1) = -\frac{(\tilde{x}_1 - \sigma^2)^2 - (\mu_4 - \sigma^4) - 4\mu_3\tilde{x}_1}{8\sigma^3} + \frac{3(\mu_4 - \sigma^4)(\tilde{x}_1^2 - \sigma^2)}{16\sigma^5}, \]
\[ \tau_1(x_1) = \frac{\tilde{x}_1^2 - \sigma^2}{2\sigma}, \]
\[ \tau_2(x_1, x_2) = -\frac{\tilde{x}_1\tilde{x}_2}{\sigma} - \frac{(\tilde{x}_1^2 - \sigma^2)(\tilde{x}_2^2 - \sigma^2)}{4\sigma^3}, \]
\[ \tau_3(x_1, x_2, x_3) = \frac{(\tilde{x}_1^2 - \sigma^2)\tilde{x}_2\tilde{x}_3 + (\tilde{x}_2^2 - \sigma^2)\tilde{x}_1\tilde{x}_3 + (\tilde{x}_3^2 - \sigma^2)\tilde{x}_1\tilde{x}_2}{2\sigma^3} + \frac{3(\tilde{x}_1^2 - \sigma^2)(\tilde{x}_2^2 - \sigma^2)(\tilde{x}_3^2 - \sigma^2)}{8\sigma^5}, \]
and
\[ \zeta_1(x_1) = \tilde{x}_1. \]

Then we have following representations.

**Lemma 2.** If \( E[|X_1|^{4+\varepsilon}] < \infty \) for some \( \varepsilon > 0 \), we have

\[
\hat{\sigma} = \sigma + n^{-1}\delta_T + n^{-2}\sum_{i=1}^{n} \tau_0(X_i) + n^{-1}\sum_{i=1}^{n} \tau_1(X_i) + n^{-2}\sum_{C_{n,2}} \tau_2(X_i, X_j) + n^{-3}\sum_{C_{n,3}} \tau_3(X_i, X_j, X_k) + o_p(n^{-3/2})
\]

and
\[ \bar{X} = \mu + n^{-1}\sum_{i=1}^{n} \zeta_1(X_i). \]

**Proof.** See Appendix.

Note that \( \delta_S = \zeta_0(x_1) = \zeta_2(x_1, x_2) = \zeta_3(x_1, x_2, x_3) = 0 \) and
\[ t^{(0)} = \sigma, \quad s^{(0)} = \mu, \quad t^{(1)} = s^{(1)} = \delta_S = 0. \]

Thus from Theorem 1, we have the asymptotic representation of the coefficient of variation \( V_n \), and from Theorem 2, we can obtain the asymptotic mean squared error of \( V_n \). The \( n^{-1} \) term of the asymptotic mean squared error of \( V_n \) is

\[ E[\eta_1^2(X_1)] = -\frac{\sigma^2}{4\mu^2} + \frac{\mu_4}{4\mu^2\sigma^2} - \frac{\mu_3}{\mu^3} + \frac{\sigma^4}{\mu^4}. \]
For \( n^{-2} \) term, using the equation (2.8), we can show that

\[
\delta^2 + 2E[\eta_0(X_1)\eta_1(X_1)] + \frac{1}{2}E[\eta_2^2(X_1, X_2)] = \frac{1}{2\mu^2} \left( \frac{31\sigma^2}{32} - \frac{3\mu_4}{16\sigma^2} + \frac{3\mu_3^2}{2\sigma^4} - \frac{\mu_6}{4\sigma^4} + \frac{15\mu_2^2}{32\sigma^6} \right) + \frac{1}{\mu^3} \left( \frac{9\mu_3}{8} - \frac{\mu_5}{4\sigma^2} + \frac{3\mu_4\mu_3}{8\sigma^4} \right) + \frac{1}{\mu^4} \left( -\frac{25\sigma^4}{2} + \frac{9\mu_4}{2} + \frac{\mu_3^2}{\sigma^2} \right) - \frac{11\sigma^2\mu_3}{\mu^5} + \frac{9\sigma^6}{\mu^6}.
\]

Here we consider the case of the normal distribution \( N(\mu, \sigma^2) \). Since \( \mu_3 = \mu_5 = 0 \), \( \mu_4 = 3\sigma^4 \) and \( \mu_6 = 15\sigma^6 \), substituting these values for (4.2) and (4.3), we have the asymptotic mean squared error

\[
AMSE(V_n) = n^{-1} \left( \frac{\sigma^2}{2\mu^2} + \frac{\sigma^4}{\mu^4} \right) + n^{-2} \left( \frac{7\sigma^2}{16\mu^2} + \frac{\sigma^4}{2\mu^4} + \frac{9\sigma^6}{\mu^6} \right).
\]

Further using Theorems 1 and 2, we can discuss the bias correction and its asymptotic mean squared error. It is easy to see that

\[
\delta = \frac{\sigma}{8\mu} - \frac{\mu_4}{8\mu^3\sigma^3} - \frac{\mu_3}{\mu^2\sigma} + \frac{\sigma^3}{\mu^3}.
\]

Using the moment method, we can obtain an estimator of \( \delta \)

\[
\hat{\delta} = \frac{\hat{\sigma}}{8\hat{\mu}} - \frac{\hat{\mu}_4}{8\hat{\mu}\hat{\sigma}^3} - \frac{\hat{\mu}_3}{\hat{\mu}^2\hat{\sigma}} + \frac{\hat{\sigma}^3}{\hat{\mu}^3}
\]

where \( \hat{\sigma}^2 = (n - 1)^{-1} \sum(X_i - \bar{X})^2 \) and \( \hat{\mu}_k = n^{-1} \sum(X_i - \bar{X})^k \) \((k = 3, 4)\). Thus we have the bias corrected estimator

\[
V_n^* = V_n - \hat{\delta}/n.
\]

Since each term of \( \hat{\delta} \) is a sum of ratio statistics again, using Theorem 1, we can obtain an asymptotic representation of \( \hat{\delta}/n \) as

\[
n^{-1}\hat{\delta} = n^{-1}\delta + n^{-2} \sum_{i=1}^{n} \varphi(X_i) + o_p(n^{-3/2})
\]

where

\[
\varphi(x) = -\frac{\bar{x}^4 - \mu_4}{8\mu\sigma^3} - \frac{\bar{x}^3 - \mu_3}{2\mu^2\sigma}
+ \left( \frac{1}{16\mu\sigma} + \frac{3\mu_4}{16\mu^5\sigma^3} + \frac{\mu_3}{4\mu^2\sigma^3} + \frac{3\sigma}{2\mu^3} \right) (\bar{x}^2 - \sigma^2)
+ \left( -\frac{\sigma}{8\mu^2} + \frac{\mu_4}{8\mu^2\sigma^3} + \frac{\mu_3}{\mu^3\sigma} - \frac{3\sigma^3}{\mu^4} \right) \bar{x}
\]
and \( \tilde{x} = x - \mu \). Thus we have

\[
V_n^* = V_n - n^{-1} \delta - n^{-2} \sum_{i=1}^{n} \phi(X_i) + o_p(n^{-3/2})
\]

and so

\[
\text{AMSE}(V_n^*) = \text{AMSE}(V_n) - n^{-2} \{ \delta^2 + 2E[\phi(X_1)\eta_1(X_1)] \} + O(n^{-3}).
\]

Using this equation, we can obtain a mean squared error of \( V_n^* \). From direct computation, we have

\[
E[\phi(X_1)\eta_1(X_1)] = -\frac{\sigma^2}{32\mu^2} + \frac{\mu_6}{16\mu^2\sigma^4} + \frac{3\mu_4^2}{32\mu^2\sigma^6} - \frac{\mu_5}{8\mu^3\sigma^2} - \frac{5\sigma^4}{8\mu^4} + \frac{9\mu_4}{8\mu^4} + \frac{\mu_3}{4\mu^4\sigma^2} - \frac{4\sigma^2\mu_3}{\mu^5} + \frac{3\sigma^6}{\mu^6}.
\]

**[Normal distribution]**

If the underlying distribution is normal, we can show that

\[
\delta = -\frac{\sigma}{4\mu} + \frac{\sigma^3}{\mu^3}
\]

and

\[
2E[\phi(X_1)\eta_1(X_1)] = -\frac{\sigma^2}{4\mu^2} + \frac{11\sigma^4}{2\mu^4} + \frac{6\sigma^6}{\mu^6}.
\]

Thus we have the asymptotic mean squared error of \( V_n^* \)

\[
\text{AMSE}(V_n^*) = \text{AMSE}(V_n) - n^{-2} \{ \delta^2 + 2E[\phi(X_1)\eta_1(X_1)] \} + O(n^{-3})
\]

\[
= n^{-1} \left( \frac{\sigma^2}{2\mu^2} + \frac{\sigma^4}{\mu^4} \right) + n^{-2} \left( \frac{5\sigma^2}{8\mu^2} - \frac{9\sigma^4}{2\mu^4} + \frac{2\sigma^6}{\mu^6} \right) + O(n^{-3}).
\]

Since

\[
\text{AMSE}(V_n) - \text{AMSE}(V_n^*) = n^{-2} \left( -\frac{3\sigma^2}{16\mu^2} + \frac{5\sigma^4}{\mu^4} + \frac{7\sigma^6}{\mu^6} \right) + O(n^{-3}),
\]

if the underlying distribution is normal and \( \sigma/\mu \geq 1/\sqrt{28} = 0.19 \), \( V_n^* \) is superior than \( V_n \) from viewpoint of unbiasedness and asymptotic mean squared error.

**[Gamma distribution]**

Similarly we can obtain the difference of the asymptotic mean squared errors \( \text{AMSE}(V_n) \) and \( \text{AMSE}(V_n^*) \) when the underlying distribution is the gamma one. The density function is

\[
f(x) = \begin{cases} 
\frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)} & x \geq 0 \\
0 & x < 0 \end{cases}
\]
for $\alpha > 0$. In this case we have that $\mu = \alpha$ and $\sigma^2 = \alpha$, and

$$AMSE(V_n) - AMSE(V^*_n) = n^{-2} \left( -\frac{3}{16\alpha} - \frac{65}{8\alpha^2} - \frac{91}{4\alpha^3} \right) + O(n^{-3}).$$

Thus if the underlying distribution is gamma, $V_n$ is always superior than $V^*_n$ from viewpoint of asymptotic mean squared error.

[Double exponential] If the density function is

$$f(x) = \frac{1}{2a} \exp \left\{ -\frac{|x - \mu|}{a} \right\},$$

where $a > 0$, we have $E(X_1) = \mu$, $\text{Var}(X_1) = 2a^2$ and

$$AMSE(V_n) - AMSE(V^*_n) = n^{-2} \frac{a^2}{\mu^2} \left( -\frac{267}{32} + \frac{44a^2}{\mu^2} + \frac{56a^4}{\mu^4} \right) + O(n^{-3}).$$

If the underlying distribution is double exponential and $\sigma/\mu = \sqrt{2a/\mu} > 1.33$, $V^*_n$ is superior than $V_n$ from viewpoint of unbiasedness and asymptotic mean squared error.

[Chi-square distribution] When the density function is

$$f(x) = \begin{cases} \frac{1}{2^{m/2}m!} x^{m/2-1} e^{-x/2} & x > 0 \\ 0 & x \leq 0 \end{cases},$$

we have $E(X_1) = m$, $\text{Var}(X_1) = 2m$ and

$$AMSE(V_n) - AMSE(V^*_n) = n^{-2} \frac{a^2}{\mu^2} \left( -\frac{3}{8m} - \frac{65}{2m^2} - \frac{91}{2m^3} \right) + O(n^{-3}).$$

Thus if the underlying distribution is chi-square, $V_n$ is always superior than $V^*_n$ from viewpoint of asymptotic mean squared error. From the above discussion, if the underlying distribution is symmetric, $V_n$ and $V^*_n$ are comparable, but if the underlying distribution is not symmetric, $V_n$ is superior than $V^*_n$ from viewpoint of asymptotic mean squared error. In the next section, we will compare $V_n$ and $V^*_n$ by simulation.

5. Simulation

In this section, we simulate the mean squared errors of the variation of coefficient based on the 1,000,000 times replications. Table 1 lists the average mean squared errors when the underlying distribution is normal, and sample sizes are 30, 50 and 100. Since the mean squared errors are $O(n^{-1})$, these estimated values are multiplied by the sample size $n$. 
When \( \theta \) is large, the estimated mean squared errors of \( V^*_n \) are smaller than those of \( V_n \). This coincides with the result of the asymptotic mean squared errors in Section 4.

Similarly, Table 2 lists the average mean squared errors when the underlying distribution is exponential, and sample sizes are 30, 50 and 100. These estimated values are also multiplied by the sample size \( n \).

The estimated mean squared errors of \( V_n \) are always smaller than those of \( V^*_n \) and this coincides the theoretical result in Section 4.

Table 3 lists the average mean squared errors when the underlying distribution is double exponential, and sample sizes are 30, 50 and 100.

When \( \theta \) is large, the estimated mean squared errors of \( V^*_n \) are smaller than those of \( V_n \). This coincides with the result of the asymptotic mean squared errors
in Section 4.

Table 4 lists the average mean squared errors when the underlying distributions are chi-square distribution with 2 (1st row) and 4 (2nd row) degrees of freedom. The sample sizes are 30, 50 and 100, and the estimated values are also multiplied by the sample size $n$.

Table 4. Estimated mean squared errors for Chi-square distribution.

<table>
<thead>
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<th>$\theta$</th>
<th>$n = 30$</th>
<th>$n = 30$</th>
<th>$n = 50$</th>
<th>$n = 50$</th>
<th>$n = 100$</th>
<th>$n = 100$</th>
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<td></td>
<td>$V_n$</td>
<td>$V^*_n$</td>
<td>$V_n$</td>
<td>$V^*_n$</td>
<td>$V^*_n$</td>
<td>$V^*_n$</td>
</tr>
<tr>
<td>1.0(2)</td>
<td>0.7913</td>
<td>1.0942</td>
<td>0.8533</td>
<td>1.1184</td>
<td>0.9139</td>
<td>1.0980</td>
</tr>
<tr>
<td>0.71(4)</td>
<td>0.3329</td>
<td>0.4125</td>
<td>0.3464</td>
<td>0.4096</td>
<td>0.3592</td>
<td>0.3990</td>
</tr>
</tbody>
</table>

The estimated mean squared errors of $V_n$ are always smaller than those of $V^*_n$ and this coincides with the theoretical result in Section 4.

Appendix

First we review the $H$-decomposition or $ANOVA$-decomposition, which is a basic tool of the studies of the analysis of variance, the jackknife inference, etc. Let $\nu(x_1, \ldots, x_r)$ be a function which is symmetric in its arguments and $E[\nu(X_1, \ldots, X_r)] = 0$. Let us define

$$\lambda_1(x_1) = E[\nu(x_1, X_2, \ldots, X_r)],$$

$$\lambda_2(x_1, x_2) = E[\nu(x_1, x_2, \ldots, X_r)] - \lambda_1(x_1) - \lambda_1(x_2), \ldots,$$

and

$$\lambda_r(x_1, x_2, \ldots, x_r) = \nu(x_1, x_2, \ldots, x_r) - \sum_{j=1}^{r-1} \sum_{C_{r,j}} \lambda_j(x_{i_1}, x_{i_2}, \ldots, x_{i_j}).$$

Then we can show that

(A.1) $E[\lambda_k(X_1, \ldots, X_k) \mid X_1, \ldots, X_{k-1}] = 0$ a.s.

and

$$\sum_{C_{n,r}} \nu(X_{i_1}, \ldots, X_{i_r}) = \sum_{k=1}^{r} \binom{n-k}{r-k} \Lambda_k,$$

where

$$\Lambda_k = \sum_{C_{n,k}} \lambda_k(X_{i_1}, \ldots, X_{i_k}).$$

The above decomposition is due to Hoeffding (1961) and called $H$-decomposition. If $\lambda_k$ satisfies the equation (A.1), using the moment evaluations of martingales (von Bahr and Esséen (1965), and Dharmadhikari et al. (1968)), we have upper bounds of the absolute moments of $\Lambda_k$ as follows.

(1) For $1 \leq q \leq 2$, if $E|\lambda_k(X_1, \ldots, X_k)|^q < \infty$, we have

(A.2) $E|\Lambda_k|^q = O(n^k)$. 

(2) For \( q \geq 2 \), if \( E|\lambda_k(X_1, \ldots, X_r)|^q < \infty \), we have
\[
E|\Delta_k|^q = O(n^{(qk)/2}).
\]

Hereafter in order to obtain evaluations of moments, we use the \( H \)-decomposition and the inequalities (A.2) and (A.3). It follows from Markov’s inequality that if
\[
R = o_p(n^{-3/2}) \quad \text{for some } \beta \geq 1 \quad \text{and } \epsilon > 0, \quad \text{we have } R = o_p(n^{-3/2}) \quad \text{for constant } c.
\]

Since we need an approximation of the product of two statistics, we prepare the following lemma. Hereafter for the purpose of simplicity, we use abbreviations \( \tau_1(i), \tau_2(i, j), \ldots \) which represent \( \tau_1(X_i), \tau_2(X_i, X_j), \ldots \).

**Lemma A.** Assume that the conditions (1.1)\textasciitilde(1.7) and (2.1)\textasciitilde(2.3) are satisfied. Then we have
\[
n^{-2} \sum_{i=1}^{n} \tau_1(i) \sum_{i=1}^{n} \zeta_1(i) = n^{-1} E[\tau_1(X)\zeta_1(X)] + n^{-2} \sum_{i=1}^{n} \{\tau_1(i)\zeta_1(i) - E[\tau_1(X)\zeta_1(X)]\}
\]
\[
+n^{-2} \sum_{C_{n,2}} \{\tau_1(i)\zeta_1(j) + \tau_1(j)\zeta_1(i)\},
\]

\[
n^{-3} \sum_{i=1}^{n} \tau_1(i) \sum_{C_{n,2}} \zeta_2(i, j)
\]

\[
= n^{-2} \sum_{i=1}^{n} E[\tau_1(X)\zeta_2(X_i, X) \mid X_i]
\]
\[
+n^{-3} \sum_{C_{n,3}} \{\tau_1(i)\zeta_2(j, k) + \tau_1(j)\zeta_2(i, k) + \tau_1(k)\zeta_2(i, j)\} + o_p(n^{-3/2}),
\]

(A.4) \[
n^{-4} \sum_{C_{n,2}} \tau_2(i, j) \sum_{C_{n,2}} \zeta_2(i, j) = o_p(n^{-3/2}),
\]

\[
n^{-4} \sum_{i=1}^{n} \tau_1(i) \sum_{C_{n,3}} \zeta_3(i, j, k) = o_p(n^{-3/2}),
\]

\[
n^{-5} \sum_{C_{n,2}} \tau_2(i, j) \sum_{C_{n,3}} \zeta_3(i, j, k) = o_p(n^{-3/2}),
\]

\[
n^{-6} \sum_{C_{n,3}} \tau_3(i, j, k) \sum_{C_{n,3}} \zeta_1(i, j, k) = o_p(n^{-3/2})
\]

and
\[
n^{-3} \left\{ \sum_{i=1}^{n} \zeta_1(i) \right\}^3 = n^{-2} \sum_{i=1}^{n} 3E[\zeta_1^2(X)]\zeta_1(i)
\]
\[
+ n^{-3} \sum_{C_{n,3}} 6\zeta_1(i)\zeta_1(j)\zeta_1(k) + o_p(n^{-3/2})
\]
where $X$ is an independent copy of $X_i$.

**Proof.** Here we will prove the equations (A.4) and (A.5). It follows from (A.3) that

$$E \left| n^{-4} \sum_{C_{n,2}} \tau_2(i, j) \sum_{C_{n,2}} \zeta_2(i, j) \right|^{2+\varepsilon/2} \leq n^{-8-2\varepsilon} \left\{ E \left| \sum_{C_{n,2}} \tau_2(i, j) \right|^{4+\varepsilon} E \left| \sum_{C_{n,2}} \zeta_2(i, j) \right|^{4+\varepsilon} \right\}^{1/2} = O(n^{-4-\varepsilon}) = O(n^{-1-3(2+\varepsilon)/2-\varepsilon/4}).$$

Thus we get the equation (A.4). From direct computation, we have

$$n^{-3} \left\{ \sum_{i=1}^{n} \zeta_1(i) \right\}^3 = n^{-3} \sum_{i=1}^{n} \zeta_1^3(i) + n^{-3} \sum_{C_{n,2}} 3\{ \zeta_1^2(i) \zeta_1(j) + \zeta_1^2(j) \zeta_1(i) \}
+n^{-3} \sum_{C_{n,3}} 6\zeta_1(i) \zeta_1(j) \zeta_1(k).$$

From the equation (A.3) we get

$$E \left| n^{-3} \sum_{i=1}^{n} \{ \zeta_1^3(i) - E[\zeta_1^3(X)] \} \right|^{(4+\varepsilon)/3} = O(n^{-3-\varepsilon}) = O(n^{-1-(3/2)(4+\varepsilon)/3-\varepsilon/2}).$$

Since $n^{-2}E[\zeta_1^3(X)] = o_p(n^{-3/2})$, we have $n^{-3} \sum_{i=1}^{n} \zeta_1^3(i) = o_p(n^{-3/2})$. Applying the $H$-decomposition to the second term, it follows from (A.3) that

$$n^{-3} \sum_{C_{n,2}} 3\{ \zeta_1^2(i) \zeta_1(j) + \zeta_1^2(j) \zeta_1(i) \} = n^{-2} \sum_{i=1}^{n} 3E[\zeta_1^3(X)] \zeta_1(i) + o_p(n^{-3/2}).$$

Thus we have the equation (A.5). We can similarly prove the others.

Using this lemma, we will obtain the asymptotic representations.

**Proof of Theorem 1.** Using Taylor expansion of $(c + x)^{-1}$, we have

$$S_n^{-1} = s_n^{-1} - s_n^{-2}(S_n - s_n) + s_n^{-3}(S_n - s_n)^2
- s_n^{-4}(S_n - s_n)^3 + (s_n + \vartheta)^{-5}(S_n - s_n)^4$$

where $0 \leq |\vartheta| \leq |S_n - s_n|$. It is easy to see that

$$-s_n^{-2}(S_n - s_n) = -n^{-1} \left( \frac{\delta s}{(s(0))^2} \right) - n^{-2} \sum_{i=1}^{n} \left\{ \frac{\zeta_0(i)}{(s(0))^2} - \frac{2s^{(1)}(i)}{(s(0))^3} \right\} - n^{-1} \sum_{i=1}^{n} \frac{\zeta_1(i)}{(s(0))^2}
- n^{-2} \sum_{C_{n,2}} \frac{\zeta_2(i, j)}{(s(0))^2} - n^{-3} \sum_{C_{n,3}} \frac{\zeta_3(i, j, k)}{(s(0))^2} + o_p(n^{-3/2}).$$
It follows from Lemma A that under the moment conditions (2.1)\~(2.3),

\[
s_n^{-3}(S_n - s_n)^2 = n^{-1}E[\xi_1^2(X)](s(0))^3 + n^{-2} \sum_{C_{n,2}} \frac{2\zeta_1(i) \zeta_1(j)}{(s(0))^3} + n^{-2} \sum_{i=1}^n \frac{\zeta_1^2(i) - E[\xi_1^2(X)] + 2\delta_S \zeta_1(i) + 2E[\xi_1(X)\xi_2(X_i, X) | X_i]}{(s(0))^3} + n^{-3} \sum_{C_{n,3}} \frac{2\{\zeta_1(i)\zeta_2(j, k) + \zeta_1(j)\zeta_2(i, k) + \zeta_1(k)\zeta_2(i, j)\}}{(s(0))^3} + o_p(n^{-3/2}).
\]

Using this representation, it follows from Lemma A that

\[
-s_n^{-4}(S_n - s_n)^3 = -s_n^{-1}(S_n - s_n)s_n^{-3}(S_n - s_n)^2 = -n^{-2} \sum_{i=1}^n 3E[\xi_1^2(X)]\zeta_1(i)(s(0))^4 - n^{-3} \sum_{C_{n,3}} \frac{6\zeta_1(i)\zeta_1(j)\zeta_1(k)}{(s(0))^4} + o_p(n^{-3/2}).
\]

Again using Lemma A, we have

\[
(S_n - s_n)^4 = o_p(n^{-3/2}).
\]

Using Markov’s inequality, we can show that

\[
P\left\{|\vartheta| > \frac{|s(0)|}{2}\right\} = o(n^{-1}) \quad \text{or} \quad P\left\{|s_n + \vartheta| \leq \frac{|s(0)|}{2}\right\} = o(n^{-1}).
\]

Since

\[
P\left\{|n^{1/2}(s_n + \vartheta)^{-5}(S_n - s_n)^4| \geq n^{-1}(\log n)^{-1}\right\} \leq P\left\{|n^{1/2}\frac{32}{(s(0))^5}(S_n - s_n)^4| \geq n^{-1}(\log n)^{-1}\right\} + P\left\{|s_n + \vartheta| \leq \frac{|s(0)|}{2}\right\} = o(n^{-1}),
\]

we have \((s_n + \vartheta)^{-5}(S_n - s_n)^4 = o_p(n^{-3/2}).\) Thus we obtain

\[
S_n^{-1} = s_n^{-1} + n^{-1}\delta_\nu + n^{-2} \sum_{i=1}^n \nu_0(i) + n^{-1} \sum_{i=1}^n \nu_1(i) + n^{-2} \sum_{C_{n,2}} \nu_2(i, j) + n^{-3} \sum_{C_{n,3}} \nu_3(i, j, k) + o_p(n^{-3/2})
\]

where

\[
\delta_\nu = \frac{E[\xi_1^2(X)]}{(s(0))^3} - \frac{\delta_S}{(s(0))^2}.
\]
\[\nu_0(x) = -\frac{\zeta_0(x)}{(s^{(0)})^2} + \left\{ \frac{2s^{(1)} + 2\delta_s}{(s^{(0)})^3} - \frac{3E[\zeta_1^2(X)]}{(s^{(0)})^4} \right\} \zeta_1(x) + \frac{\zeta_1^2(x) - E[\zeta_1^2(X)] + 2E[\zeta_1(X)\zeta_2(x, X)]}{(s^{(0)})^3},\]

\[\nu_1(x) = -\frac{\zeta_1(x)}{(s^{(0)})^2},\]

\[\nu_2(x, y) = -\frac{\zeta_2(x, y)}{(s^{(0)})^2} + \frac{2\zeta_1(x)\zeta_1(y)}{(s^{(0)})^3}\]

and

\[\nu_3(x, y, z) = -\frac{\zeta_3(x, y, z)}{(s^{(0)})^2} - \frac{6\zeta_1(x)\zeta_1(y)\zeta_1(z)}{(s^{(0)})^4} + \frac{2\{\zeta_1(x)\zeta_2(y, z) + \zeta_1(y)\zeta_2(x, z) + \zeta_1(z)\zeta_2(x, y)\}}{(s^{(0)})^3}.\]

Similarly, using Lemma A, we can show that

\[T_n s_n^{-1}\]

\[= \frac{t_n}{s_n} + n^{-1} \frac{\delta_T}{s^{(0)}} + n^{-2} \sum_{i=1}^{n} \left\{ \tau_0(i) - \frac{s^{(1)} \tau_1(i)}{(s^{(0)})^2} \right\} + n^{-1} \sum_{i=1}^{n} \frac{\tau_1(i)}{s^{(0)}} + n^{-3} \sum_{C_{n,2}} \tau_2(i, j) + o_p(n^{-3/2}),\]

\[T_n n^{-1} \delta_\nu\]

\[= n^{-1} t^{(0)} \delta_\nu + n^{-2} \sum_{i=1}^{n} \delta_\nu \tau_1(i) + o_p(n^{-3/2}),\]

\[T_n n^{-2} \sum_{i=1}^{n} \nu_0(i)\]

\[= n^{-2} \sum_{i=1}^{n} t^{(0)} \nu_0(i) + o_p(n^{-3/2}),\]

\[T_n n^{-1} \sum_{i=1}^{n} \nu_1(i)\]

\[= n^{-1} E[\tau_1(X)\nu_1(X)] + n^{-3} \sum_{C_{n,2}} \{ \tau_1(i) \nu_1(j) + \tau_1(j) \nu_1(i) \} + n^{-2} \sum_{C_{n,2}} \{ \nu_1(i) \tau_2(j, k) + \nu_1(j) \tau_2(i, k) + \nu_1(k) \tau_2(i, j) \} + o_p(n^{-3/2}),\]

\[T_n n^{-2} \sum_{C_{n,2}} \nu_2(i, j)\]
Using Lemma A, we can show that
\[
\frac{1}{n} \sum_{i=1}^{n} \{Y_i - \bar{Y}\} \leq \sigma_y + n^{-1} \sum_{i=1}^{n} \{Y_i^2 - \sigma_y^2\} + n^{-2} \sum_{j=1}^{n-1} \{2Y_iY_j\} + o_p(n^{-3/2}),
\]
and
\[
\frac{1}{n} \sum_{i=1}^{n} \{Z_i - \bar{Z}\} \leq \sigma_z + n^{-1} \sum_{i=1}^{n} \{Z_i^2 - \sigma_z^2\} + n^{-2} \sum_{j=1}^{n-1} \{2Z_iZ_j\} + o_p(n^{-3/2}).
\]

Combining the above equations, we have the desired result.

**Proof of Lemma 1.** Applying $H$-decomposition, we have

\[
(n-1)^{-1} \sum_{i=1}^{n} (Y_i - \bar{Y})^2 = \sigma_y^2 + n^{-1} \sum_{i=1}^{n} \{Y_i^2 - \sigma_y^2\} + n^{-2} \sum_{j=1}^{n-1} \{2Y_iY_j\} + o_p(n^{-3/2}),
\]

\[
(n-1)^{-1} \sum_{i=1}^{n} (Z_i - \bar{Z})^2 = \sigma_z^2 + n^{-1} \sum_{i=1}^{n} \{Z_i^2 - \sigma_z^2\} + n^{-2} \sum_{j=1}^{n-1} \{2Z_iZ_j\} + o_p(n^{-3/2}).
\]

Using Lemma A, we can show that

\[
(n-1)^{-2} \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \sum_{i=1}^{n} (Z_i - \bar{Z})^2 = \sigma_y^2 \sigma_z^2 + n^{-1} \delta_{\psi} + n^{-2} \sum_{i=1}^{n} \psi_0(X_i) + n^{-1} \sum_{i=1}^{n} \psi_1(X_i) + n^{-2} \sum_{i=1}^{n} \psi_2(X_i, X_j) + n^{-3} \sum_{i=1}^{n} \psi_3(X_i, X_j, X_k) + o_p(n^{-3/2})
\]

where

\[
\delta_{\psi} = E[Y_1^2 Z_1^2] - \sigma_y^2 \sigma_z^2,
\]

\[
\psi_0(x_1) = (y_1^2 - \sigma_y^2)(z_1^2 - \sigma_z^2) - \delta - E[Y_2^2 Z_2] z_1 - E[Y_2 Z_2^2] y_1,
\]

\[
\psi_1(x_1) = \sigma_z^2 (y_1^2 - \sigma_y^2) + \sigma_y^2 (z_1^2 - \sigma_z^2),
\]

\[
\psi_2(x_1, x_2) = (y_1^2 - \sigma_y^2)(z_2^2 - \sigma_z^2) + (y_2^2 - \sigma_y^2)(z_1^2 - \sigma_z^2) - \sigma_z^2 y_1 y_2 - \sigma_y^2 z_1 z_2
\]

and
\[\psi_3(x_1, x_2, x_3) = -\{(y_1^2 - \sigma_y^2)z_2z_3 + (y_2^2 - \sigma_y^2)z_1z_3 + (y_3^2 - \sigma_y^2)z_1z_2
+ (z_1^2 - \sigma_z^2)y_2y_3 + (z_2^2 - \sigma_z^2)y_1y_3 + (z_3^2 - \sigma_z^2)y_1y_2\}.\]

Using Taylor expansion of \(\sqrt{x}\), we can show that

\[S_n = \sigma_y\sigma_z + n^{-1}\left\{\frac{\delta_\psi}{2\sigma_y\sigma_z} - \frac{E[\psi_1^2(X)]}{8\sigma_y^3\sigma_z^3}\right\}
+n^{-2}\sum_{i=1}^n \left\{\frac{\psi_0(i)}{2\sigma_y\sigma_z} - \frac{\psi_1^2(i) - E[\psi_1^2(X)] + 2\delta_\psi\psi_1(i) + 2E[\psi_1(X)\psi_2(i, X)]}{8\sigma_y^3\sigma_z^2}\right\}
+n^{-1}\sum_{i=1}^n \frac{\psi_1(i)}{2\sigma_y\sigma_z} + n^{-2}\sum_{C_{n,2}} \left\{\frac{\psi_2(i, j)}{2\sigma_y\sigma_z} - \frac{\psi_1(i)\psi_1(j)}{4\sigma_y^3\sigma_z^3}\right\}
+n^{-3}\sum_{C_{n,3}} \left\{\frac{\psi_3(i, j, k)}{2\sigma_y\sigma_z} - \frac{\psi_1(i)\psi_2(j, k) + \psi_1(j)\psi_2(i, k) + \psi_1(k)\psi_2(i, j)}{4\sigma_y^3\sigma_z^3}\right\}
+ \frac{3\psi_1(i)\psi_1(j)\psi_1(k)}{8\sigma_y^5\sigma_z^5} + o_p(n^{-3/2})\]

where \(X\) is an independent copy of \(X_i\). Thus we have the desired result.

**Proof of Lemma 2.** The representation of \(X\) is easy. From the proof of Lemma 1, we have

\[\hat{\sigma}^2 = (n - 1)^{-1}\sum_{i=1}^n (X_i - \bar{X})^2
= \sigma^2 + n^{-1}\sum_{i=1}^n \{X_i^2 - \sigma^2\} + n^{-2}\sum_{C_{n,2}} \{-2X_iX_j\} + o_p(n^{-3/2}).\]

Using Taylor expansion of \(\sqrt{x}\) and Lemma A, we can show the equation (4.1).

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**References**


