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CHARACTERIZATION OF BALANCED FRACTIONAL 2^m FACTORIAL DESIGNS OF RESOLUTION $R^*(\{1\}|3)$ AND GA-OPTIMAL DESIGNS

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In this paper, based on the assumption that the four-factor and higher-order interactions are to be negligible, we consider a balanced fractional 2^m factorial design derived from a simple array such that all the main effects are estimable, i.e., resolution $R^*(\{1\}|3)$. In this situation, using the algebraic structure of the triangular multidimensional partially balanced association scheme and a matrix equation, we can get designs of four types of resolutions: the first is of resolution $R(\{1\}|3)$, the second is of resolution $R(\{0,1\}|3)$, the third is of resolution $R(\{1,2\}|3)$, i.e., resolution VI, and the last is of resolution $R(\{0,1,2\}|3)$, i.e., resolution VI. This paper gives the characterization of designs mentioned above, and also it gives optimal designs with respect to the generalized A-optimality criterion for $6 \leq m \leq 8$ when the number of assemblies is less than the number of non-negligible factorial effects.

Key words and phrases: Association algebra, BFF designs, estimable parametric functions, GA-optimality criterion, resolution, simple arrays.

1. Introduction

The concept of a balanced array (B-array) was first introduced by Chakravarti (1956) as a generalization of an orthogonal array. Under certain conditions, a B-array of strength 2ℓ and two symbols turns out to be a balanced fractional 2^m factorial (2^m-BFF) design of resolution $2\ell+1$ (e.g., Srivastava (1970), and Yamamoto *et al.* (1975)), where $2\ell \leq m$. The characteristic roots of the information matrix of a 2^m-BFF design of resolution V (i.e., $\ell = 2$) were obtained by Srivastava and Chopra (1971). By applying the algebraic structure of the triangular multidimensional partially balanced (TMDPB) association scheme, their results were extended to 2^m-BFF designs of resolution $2\ell+1$ by Yamamoto *et al.* (1976).

As the extension of the concept of resolution, Yamamoto and Hyodo (1984) discussed the extended concept of resolution for 2^m fractions.

Definition 1.1. *Under the assumption that the $(\ell+1)$-factor and higher-order interactions are to be negligible, if the p_1-factor, the p_2-factor, ..., and the p_f-factor interactions are estimable, where $0 \leq p_1 < p_2 < \cdots < p_f \leq \ell$, then a*

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design is said to be of resolution $\mathrm{R}^*(\{p_1, p_2, \dots, p_f\}|\ell)$*, and in addition, if the remaining factorial effects are confounded with each other, then a design is said to be of resolution* $\mathrm{R}(\{p_1, p_2, \dots, p_f\}|\ell)$*. In particular, when* $p_i = i-1\,(1 \le i \le f)$ *and* $f = \ell+1$*, it is of resolution* $2\ell+1$*, and when* $p_i = i-1\,(1 \le i \le f)$ *and* $f = \ell$ *or* $p_i = i\,(1 \le i \le f)$ *and* $f = \ell - 1$*, it is of resolution* 2ℓ*.*

Note that if a design is of resolution $\mathrm{R}^*(\{p_1, p_2, \dots, p_f\}|\ell)$, then it is also of resolution $\mathrm{R}(\{q_1, q_2, \dots, q_g\}|\ell)$, where $0 \le q_1 < q_2 < \cdots < q_g \le \ell$ and $\{q_1, q_2, \dots, q_g\} \supset \{p_1, p_2, \dots, p_f\}$. For example, a resolution $\mathrm{R}^*(\{1\}|3)$ design is of resolution $\mathrm{R}(\{1\}|3), \mathrm{R}(\{0,1\}|3), \mathrm{R}(\{1,2\}|3), \mathrm{R}(\{1,3\}|3), \mathrm{R}(\{0,1,2\}|3), \mathrm{R}(\{0,1,3\}|3), \mathrm{R}(\{1,2,3\}|3)$ or $\mathrm{R}(\{0,1,2,3\}|3)$. Here a resolution $\mathrm{R}(\{0,1,2\}|3)$ or $\mathrm{R}(\{1,2\}|3)$ design is of resolution VI, and a resolution $\mathrm{R}(\{0,1,2,3\}|3)$ one is of resolution VII.

Some estimable parametric functions of the interesting factorial effects have been studied by several authors (e.g., Hyodo (1989), and Kuwada and Yanai (1998)). Especially using the properties of the TMDPB association algebra and a matrix equation, Ghosh and Kuwada (2001) obtained some estimable parametric functions for 2^m-BFF designs. As a generalization of the A-optimality criterion, Kuwada *et al.* (2002) have introduced the generalized A-optimality (GA-optimality) criterion and they have also given GA-optimal 2^m-BFF designs of resolution $\mathrm{R}^*(\{0,1\}|3)$ derived from simple arrays for $6 \le m \le 8$. Here a simple array is a B-array of full strength and index set $\{\lambda_i \mid 0 \le i \le m\}$, i.e., a B-array of strength m and size N having m constraints, two symbols and index set $\{\lambda_i\}$, and it is written as $\mathrm{SA}(m; \{\lambda_i\})$ for brevity. A necessary and sufficient condition for a B-array of strength 2ℓ to be a 2^m-BFF design of resolution $\mathrm{R}(\{1, \dots, \ell-1\}|\ell)$, i.e., resolution 2ℓ, was given by Shirakura (1980).

In this paper, using the properties of the TMDPB association algebra and the matrix equation, we characterize 2^m-BFF designs of resolution $\mathrm{R}^*(\{1\}|3)$ derived from simple arrays, and we give optimal designs with respect to the GA-optimality criterion for $6 \le m \le 8$ when the number of assemblies (or treatment combinations) is less than the number of non-negligible factorial effects.

2. Preliminaries

Consider a fractional 2^m factorial design, T, say, with N assemblies, where the four-factor and higher-order interactions are assumed to be negligible and $m \ge 6$. Then the $1 \times \nu_3$ vector of non-negligible factorial effects is given by $\boldsymbol{\Theta}' = (\boldsymbol{\theta}_0'; \boldsymbol{\theta}_1'; \boldsymbol{\theta}_2'; \boldsymbol{\theta}_3')$, where A' is the transpose of a matrix A, $\nu_3 = d_0 + d_1 + d_2 + d_3$, $d_p = \binom{m}{p}$, $\boldsymbol{\theta}_0' = \{\theta_\phi\}$, $\boldsymbol{\theta}_1' = \{\theta_t \mid 1 \le t \le m\}$, $\boldsymbol{\theta}_2' = \{\theta_{t_1 t_2} \mid 1 \le t_1 < t_2 \le m\}$ and $\boldsymbol{\theta}_3' = \{\theta_{t_1 t_2 t_3} \mid 1 \le t_1 < t_2 < t_3 \le m\}$. Here θ_ϕ, θ_t, $\theta_{t_1 t_2}$ and $\theta_{t_1 t_2 t_3}$ are the general mean, the main effect of the t-th factor, the two-factor interaction of the t_1-th and t_2-th ones, and the three-factor one of the t_1-th, t_2-th and t_3-th ones, respectively. Thus the linear model based on T is given by

$$\varepsilon[\boldsymbol{y}(T)] = E_T \boldsymbol{\Theta}, \qquad \mathrm{Var}[\boldsymbol{y}(T)] = \sigma^2 I_N,$$

where $\boldsymbol{y}(T)$, E_T and I_p are an $N \times 1$ vector of observations based on T, the design matrix of size $N \times \nu_3$ whose elements are either 1 or -1 and the identity matrix of order p, respectively. Here $\varepsilon[\boldsymbol{y}]$ denotes the expected value of a random vector $\boldsymbol{y}$, and σ^2 may or may not be known. Then the normal equations for estimating $\boldsymbol{\Theta}$ are given by

$$M_T \hat{\boldsymbol{\Theta}} = E_T' \boldsymbol{y}(T), \tag{2.1}$$

where $M_T(= E_T' E_T)$ is the information matrix of order ν_3.

Let $A_\alpha^{(u,v)}$ and $D_\alpha^{(u,v)}$ $(\alpha \leq u,\ v \leq 3;\ 0 \leq \alpha \leq 3)$ be the $d_u \times d_v$ local association matrices and the $\nu_3 \times \nu_3$ ordered association matrices of the TMDPB association scheme, respectively. Further let $A_\beta^{\#(u,v)}$ and $D_\beta^{\#(u,v)}$ $(\beta \leq u,\ v \leq 3;\ 0 \leq \beta \leq 3)$ be respectively the matrices of size $d_u \times d_v$ and of order ν_3, where the relationship between $A_\alpha^{(u,v)}$ and $A_\beta^{\#(u,v)}$, and $D_\alpha^{(u,v)}$ and $D_\beta^{\#(u,v)}$ are given by

(2.2a)

$$A_\alpha^{(u,v)} \left(= A_\alpha^{(v,u)\prime}\right) = \sum_{\beta=0}^{u} z_{\beta\alpha}^{(u,v)} A_\beta^{\#(u,v)}, \quad D_\alpha^{(u,v)} \left(= D_\alpha^{(v,u)\prime}\right) = \sum_{\beta=0}^{u} z_{\beta\alpha}^{(u,v)} D_\beta^{\#(u,v)}$$

$$\text{for} \quad \alpha \leq u \leq v \leq 3 \quad \text{and} \quad 0 \leq \alpha \leq 3,$$

(2.2b)

$$A_\beta^{\#(u,v)} \left(= A_\beta^{\#(v,u)\prime}\right) = \sum_{\alpha=0}^{u} z_{(u,v)}^{\beta\alpha} A_\alpha^{(u,v)}, \quad D_\beta^{\#(u,v)} \left(= D_\beta^{\#(v,u)\prime}\right) = \sum_{\alpha=0}^{u} z_{(u,v)}^{\beta\alpha} D_\alpha^{(u,v)}$$

$$\text{for} \quad \beta \leq u \leq v \leq 3 \quad \text{and} \quad 0 \leq \beta \leq 3,$$

$$z_{\beta\alpha}^{(u,v)} = \sum_{b=0}^{\alpha} \left\{ (-1)^{\alpha-b} \binom{u-\beta}{b} \binom{u-b}{u-\alpha} \times \binom{m-u-\beta+b}{b} \sqrt{\binom{m-u-\beta}{v-u}\binom{v-\beta}{v-u}} \Big/ \binom{v-u+b}{b} \right\}$$

$$\text{for} \quad u \leq v,$$

$$z_{(u,v)}^{\beta\alpha} = \phi_\beta z_{\beta\alpha}^{(u,v)} \Big/ \left\{ \binom{m}{u}\binom{u}{\alpha}\binom{m-u}{v-u+\alpha} \right\} \quad \text{for} \quad u \leq v,$$

(2.3)

$$\phi_\beta = \binom{m}{\beta} - \binom{m}{\beta-1}$$

(see Shirakura and Kuwada (1976)), and Yamamoto *et al.* (1976)). The properties of $A_\beta^{\#(u,v)}$'s and $D_\beta^{\#(u,v)}$'s are cited in the following:

(2.4)

$$\sum_{\beta=0}^{u} A_\beta^{\#(u,u)} = I_{d_u}, \quad A_\beta^{\#(u,w)} A_\gamma^{\#(w,v)} = \delta_{\beta\gamma} A_\beta^{\#(u,v)}, \quad \text{rank}\left\{A_\beta^{\#(u,v)}\right\} = \phi_\beta,$$

$$\sum_{u=0}^{3}\sum_{\beta=0}^{u} D_\beta^{\#(u,u)} = I_{\nu_3}, \quad D_\beta^{\#(u,w)} D_\gamma^{\#(s,v)} = \delta_{ws}\delta_{\beta\gamma} D_\beta^{\#(u,v)}, \quad \text{rank}\left\{D_\beta^{\#(u,v)}\right\} = \phi_\beta$$

(see Yamamoto *et al.* (1976)), where δ_{pq} is the Kronecker delta.

Let $\mathcal{A} = [D_\alpha^{(u,v)} \mid \alpha \le u, v \le 3; 0 \le \alpha \le 3]$, where $[D_\alpha^{(u,v)}]$ denotes the algebra generated by the linear closure of these matrices indicated in the bracket [ ]. Note that $\mathcal{A}$ is called the TMDPB association algebra. Then from (2.2a,b), we get $\mathcal{A} = [D_\beta^{\#(u,v)} \mid \beta \le u,\ v \le 3;\quad 0 \le \beta \le 3]$. Further let $\mathcal{A}_\beta = [D_\beta^{\#(u,v)} \mid \beta \le u,\ v \le 3]$ for $0 \le \beta \le 3$. Then the following is a special case due to Yamamoto *et al.* (1976):

Proposition 2.1. (I) *The TMDPB association algebra $\mathcal{A}$ generated by $D_\beta^{\#(u,v)} (\beta \le u,\ v \le 3;\ 0 \le \beta \le 3)$ is semisimple and completely reducible matrix algebra containing I_{ν_3}.*
(II) *$\mathcal{A}_\beta$ are the minimal two-sided ideals of $\mathcal{A}$.*
(III) *$\mathcal{A}$ is decomposed into the direct sum of four two-sided ideals $\mathcal{A}_\beta$ of $\mathcal{A}$.*
(IV) *$\mathcal{A}_\beta$ have $D_\beta^{\#(u,v)}$ as their bases, and each ideal $\mathcal{A}_\beta$ is isomorphic to the complete $(4-\beta)\times(4-\beta)$ matrix algebra with multiplicity ϕ_β.*

Let T be a 2^m-BFF design derived from an SA$(m;\{\lambda_i\})$. Then $N = \sum_{i=0}^{m}\binom{m}{i}\lambda_i$, and the information matrix M_T associated with T is given by

$$M_T = \sum_{u=0}^{3}\sum_{v=0}^{3}\sum_{\alpha=0}^{\min(u,v)} \gamma_{|v-u|+2\alpha} D_\alpha^{(u,v)} = \sum_{u=0}^{3}\sum_{v=0}^{3}\sum_{\beta=0}^{\min(u,v)} \kappa_\beta^{u-\beta,v-\beta} D_\beta^{\#(u,v)},$$

where

$$\gamma_i = \sum_{j=0}^{m}\sum_{p=0}^{i}(-1)^p\binom{i}{p}\binom{m-i}{j-i+p}\lambda_j \qquad \text{for} \quad 0 \le i \le 6,$$

$$\kappa_\beta^{u,v}\left(=\kappa_\beta^{v,u}\right) = \sum_{\alpha=0}^{\beta+u}\gamma_{v-u+2\alpha} z_{\beta\alpha}^{(\beta+u,\,\beta+v)} \qquad \text{for} \quad 0 \le u \le v \le 3-\beta$$

(see Yamamoto *et al.* (1976)). Here the relationship between $\kappa_\beta^{u,v}$'s and λ_i's are given in Appendix A. Thus from Proposition 2.1, M_T associated with T is isomorphic to $\| \kappa_\beta^{u,v} \|$ ($= K_\beta$, say) of order $(4-\beta)$ for $0 \le \beta \le 3$, i.e., there exists an orthogonal matrix Q of order ν_3 such that $Q' M_T Q = \text{diag}[K_0; K_1, \ldots, K_1; K_2, \ldots, K_2; K_3, \ldots, K_3]$, where the multiplicities of K_β are ϕ_β. The matrices K_β are called the irreducible representations of M_T with respect to the ideals $\mathcal{A}_\beta$.

Remark 2.1. The first, the second, ... , and the last rows (and columns) of $K_\beta(0 \le \beta \le 3)$ correspond to the β-factor interactions, the $(\beta+1)$-factor ones, ..., and the three-factor ones, respectively.

PROPOSITION 2.2 (see Hyodo (1989)). *Let T be an* $\mathrm{SA}(m; \{\lambda_i\})$. *Then*
(I) $\mathrm{rank}\{K_\beta\} = r_\beta(0 \le \beta \le 3)$ *if and only if exactly r_β of the indices $\lambda_i(\beta \le i \le m-\beta)$ are nonzero, where $r_\beta < 4-\beta$,*
(II) *if* $\mathrm{rank}\{K_\beta\} = r_\beta(\le 4-\beta)$, *then the first r_β rows* (*and columns*) *of K_β are linearly independent.*

From Proposition 2.1, we have the following (see Yamamoto *et al.* (1976)):

PROPOSITION 2.3. *Let T be an $SA(m; \{\lambda_i\})$. Then the information matrix M_T associated with T is nonsingular, i.e., T is of resolution* VII, *if and only if every $K_\beta(0 \le \beta \le 3)$ are positive definite.*

The following is due to Shirakura and Kuwada (1975):

PROPOSITION 2.4. *Let T be an* $\mathrm{SA}(m; \{\lambda_i\})$, *and further let $\bar{T}$ be the complementary array of T, i.e., $\bar{T}$ is the* $\mathrm{SA}(m; \{\bar{\lambda}_i\})$, *where $\bar{\lambda}_i = \lambda_{m-i}$ for $0 \le i \le m$. Then we have $\bar{K}_\beta = \Delta_\beta K_\beta \Delta_\beta$ for $0 \le \beta \le 3$, where $\bar{K}_\beta$ are the irreducible representations of $M_{\bar{T}}$ with respect to the ideals $\mathcal{A}_\beta$ and Δ_β are the $(4-\beta) \times (4-\beta)$ diagonal matrices whose (i,i) elements are $(-1)^i$ for $0 \le i \le 3-\beta$.*

3. Estimable parametric functions

In this section, attention is focused on obtaining 2^m-BFF designs of resolution $\mathrm{R}^*(\{1\}|3)$, which are derived from simple arrays. A parametric function $C\boldsymbol{\Theta}$ of $\boldsymbol{\Theta}$ is estimable for some matrix C of order ν_3 if and only if there exists a matrix X of order ν_3 such that $XM_T = C$ (e.g., Yamamoto and Hyodo (1984)). If $C\boldsymbol{\Theta}$ is estimable, then its unbiased estimator is given by $C\hat{\boldsymbol{\Theta}}$, and $\mathrm{Var}[C\hat{\boldsymbol{\Theta}}] = \sigma^2 X M_T X'$, where $\hat{\boldsymbol{\Theta}}$ is a solution of the equations (2.1). Furthermore since M_T belongs to $\mathcal{A}$, we impose some restrictions on C such that it belongs to $\mathcal{A}$, and hence X also belongs to $\mathcal{A}$, i.e.,

$$C = g_0^{0,0} D_0^{\#(0,0)} + \sum_{u=2}^{3} \left(g_0^{0,u} D_0^{\#(0,u)} + g_0^{u,0} D_0^{\#(u,0)} \right) + D_0^{\#(1,1)} + D_1^{\#(1,1)}$$
$$+ \sum_{u=2}^{3} \sum_{v=2}^{3} \sum_{\beta=0}^{\min(u,v)} g_\beta^{u-\beta,v-\beta} D_\beta^{\#(u,v)},$$
$$X = \sum_{u=0}^{3} \sum_{v=0}^{3} \sum_{\beta=0}^{\min(u,v)} x_\beta^{u-\beta,v-\beta} D_\beta^{\#(u,v)},$$

where $g_\beta^{u,v}$'s and $x_\beta^{u,v}$'s are some constants. Then from Proposition 2.1, C and X are isomorphic to $\| g_\beta^{u,v} \|$ ($= \mathit{\Gamma}_\beta$, say) and $\| x_\beta^{u,v} \|$ ($= \chi_\beta$, say), respectively, where $g_0^{1,1} = g_1^{0,0} = 1$ and $g_0^{u,1} = g_0^{1,u} = g_1^{0,v} = g_1^{v,0} = 0$ for $u = 0, 2, 3$ and

$v = 1, 2$, and both $\mathit{\Gamma}_\beta$ and χ_β are of order $(4-\beta)$ for $0 \le \beta \le 3$. Hence $XM_T = C$ is isomorphic to $\chi_\beta K_\beta = \mathit{\Gamma}_\beta$. Thus in this paper, one of the aim is to choose the constants $g_\beta^{u,v}$'s of $\mathit{\Gamma}_\beta$ under some restrictions on the estimability of non-negligible factorial effects.

At the beginning, we consider a matrix equation $ZL = H$ with parameter matrix Z of order n, where $L = \| L_{ij} \|$ and $H = \| H_{ij} \|$ $(i, j = 1, 2, 3)$ are the positive semidefinite matrix of order n with $\mathrm{rank}\{L\} = \mathrm{rank}\left\{\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}\right\} = n_1 + n_2 (\ge 1)$ and some matrix of order n with $H_{11} = I_{n_1}$, $H_{12} = H'_{21} = O_{n_1 \times n_2}$ and $H_{13} = H'_{31} = O_{n_1 \times n_3}$, respectively. Here L_{ij} and H_{ij} are of size $n_i \times n_j$, $n_1 + n_2 + n_3 = n$, and $O_{p \times q}$ is the zero matrix of size $p \times q$. The matrix equation $ZL = H$ has a solution if and only if $\mathrm{rank}\{L'\} = \mathrm{rank}\{L'; H'\}$. Thus we have the following (see Ghosh and Kuwada (2001)):

Lemma 3.1. *The matrix equation $ZL = H$ has a solution if and only if*
(I) $n_3 = 0$, *where H_{22} is arbitrary, or*
(II) $n_3 \ge 1$ *and in addition*
(i) *when $n_2 = 0$, $L_{33} = O_{n_3 \times n_3}$, and furthermore $H_{33} = O_{n_3 \times n_3}$,*
(ii) *when $n_2 \ge 1$, there exists a matrix W of size $n_3 \times n_2$ such that $(L_{31}; L_{32}; L_{33}) = W(L_{21}; L_{22}; L_{23})$, and furthermore $H'_{23} = WH'_{22}$ and $H'_{33} = WH'_{32}$, where H_{22} and H_{32} are arbitrary.*

In Lemma 3.1, the matrix equation $ZL = H$ has a solution Z such that $Z = HL^{-1}$ for (I), $\begin{pmatrix} L_{11}^{-1} & Z_{13} \\ 0 & Z_{33} \end{pmatrix}$ for (II)(i), where Z_{i3} $(i = 1, 3)$ are arbitrary, and $\left(\begin{pmatrix} I_{n_1} & 0 \\ 0 & H_{22} \\ 0 & H_{32} \end{pmatrix} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}^{-1} - \begin{pmatrix} 0 & Z_{13}W \\ 0 & Z_{23}W \\ 0 & Z_{33}W \end{pmatrix}; \begin{pmatrix} Z_{13} \\ Z_{23} \\ Z_{33} \end{pmatrix}\right)$ for (II)(ii), where Z_{i3} $(i = 1, 2, 3)$ are arbitrary. Since $\mathrm{rank}\{L\} = n_1 + n_2$, $H_{11} = I_{n_1}$, $H_{12} = H'_{21} = O_{n_1 \times n_2}$ and $H_{13} = H'_{31} = O_{n_1 \times n_3}$, we have $n_1 \le \mathrm{rank}\{H\} \le n_1 + n_2$. Furthermore, since H_{22} (if $n_2 \ge 1$) is arbitrary, we can get H_{22} with $\mathrm{rank}\{H_{22}\} = n_2$, and hence $\mathrm{rank}\{H\} = n_1 + n_2$. Thus if $n_2 \ge 1$ and $n_3 \ge 1$, then there exists a matrix U of size $n_3 \times n_2$ such that $H_{32} = UH_{22}$. While from Lemma 3.1, if $\mathrm{rank}\{K_\beta\} = 4 - \beta$ for some β $(0 \le \beta \le 3)$, then we put $\mathit{\Gamma}_\beta = I_{4-\beta}$, and if $\mathrm{rank}\{K_0\} = 3$, then we put $g_0^{0,2} = g_0^{2,0} = 0$, where $g_0^{0,0} \ne 0$ and $g_0^{2,2} \ne 0$.

Let $K_0^* = PK_0P'$ and $K_\gamma^* = K_\gamma$ $(1 \le \gamma \le 3)$, where $P = \mathrm{diag}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; I_2\right]$. Then applying Lemma 3.1 to the matrix equations $\chi_\beta^* K_\beta^* = \mathit{\Gamma}_\beta^*$ $(0 \le \beta \le 3)$ with parameter matrices χ_β^*, where $\chi_0^* = P\chi_0P'$, $\mathit{\Gamma}_0^* = P\mathit{\Gamma}_0P'$, $\chi_\gamma^* = \chi_\gamma$ and $\mathit{\Gamma}_\gamma^* = \mathit{\Gamma}_\gamma$ $(1 \le \gamma \le 3)$, we have the following:

Lemma 3.2. *Let T be an $\mathrm{SA}(m; \{\lambda_i\})$. Then a necessary condition for the main effects to be estimable is that at least three of λ_i $(0 \le i \le m)$ are nonzero and in addition at least two of these suffixes are greater than or equal to 1 and less than or equal to $m - 1$.*

The proof will be given in Appendix B.

Note that from Lemma 3.2, if T is of resolution $\mathrm{R}^*(\{1\}|3)$, then $\mathrm{rank}\{K_0^*\} \geq 3$ and $\mathrm{rank}\{K_1^*\} \geq 2$. While from Proposition 2.2 and Appendix A, if T is of resolution $\mathrm{R}^*(\{3\}|3)$, then at least one of λ_i $(3 \leq i \leq m-3)$ is nonzero, and hence $\mathrm{rank}\{K_3^*\} = 1$ and $\mathrm{rank}\{K_2^*\} \geq 1$. Furthermore if $\mathrm{rank}\{K_2^*\} = 1$, then from Lemma 3.1, the three-factor interactions are confounded with the two-factor ones. Thus we have $\mathrm{rank}\{K_2^*\} = 2$ and $\mathrm{rank}\{K_1^*\} \geq 2$. Moreover, if $\mathrm{rank}\{K_1^*\} = 2$, then the three-factor interactions are confounded with the main effects and the two-factor ones, and hence $\mathrm{rank}\{K_1^*\} = 3$ and $\mathrm{rank}\{K_0^*\} \geq 3$. Similarly if $\mathrm{rank}\{K_0^*\} = 3$, then the three-factor interactions are confounded with the general mean, the main effects and the two-factor ones, and hence $\mathrm{rank}\{K_0^*\} = 4$. Therefore from Proposition 2.3, if T is of resolution $\mathrm{R}^*(\{3\}|3)$, then it is of resolution VII. This implies that if T is of resolution $\mathrm{R}(S \cup \{3\}|3)$, where $S \subset \{0,1,2\}$, then it is of resolution VII. Thus from Proposition 2.3, a resolution $\mathrm{R}^*(\{1\}|3)$ design with $\det(M_T) = 0$, i.e., $\det(K_\beta^*) = 0$ for some $\beta\,(0 \leq \beta \leq 3)$, is of resolution $\mathrm{R}(\{1\}|3), \mathrm{R}(\{0,1\}|3), \mathrm{R}(\{1,2\}|3)$ or $\mathrm{R}(\{0,1,2\}|3)$.

LEMMA 3.3. *Let T be an* $\mathrm{SA}(m;\{\lambda_i\})$ *with* $\det(M_T) = 0$. *Then a necessary condition for T to be a* 2^m-BFF *design of resolution* $\mathrm{R}^*(\{1\}|3)$ *is the following*:

(I) *if* $\mathrm{rank}\{K_0^*\} = 3$, *then there exist* $\lambda_i \neq 0$ $(i = p,q,r;\ 0 \leq p < q < r \leq m)$ *such that* $(m-2p)(m-2q)+(m-2q)(m-2r)+(m-2r)(m-2p)+(3m-2)(= \tilde{w}_0$, *say*$) = 0$, *and* $\lambda_j = 0\,(j \neq p,q,r;\ 0 \leq j \leq m)$, *and furthermore the last row of* K_0^* *is expressed by the sum of* $-\{(m-2p)(m-2q)(m-2r)+m(3m-2p-2q-2r)\}/\sqrt{6m(m-1)(m-2)}(= w_0$, *say*$)$ *times the second one of* K_0^* *and of* $-(3m-2p-2q-2r)/\sqrt{3(m-2)}(= w_0^*$, *say*$)$ *times the third,*

(II) *if* $\mathrm{rank}\{K_1^*\} = 2$, *then there exist* $\lambda_i \neq 0 (i = s,t; 1 \leq s < t \leq m-1)$ *such that* $(m-2s)(m-2t)+(m-2)(= \tilde{w}_1$, *say*$) = 0$, *and* $\lambda_j = 0$ $(j \neq s,t; 1 \leq j \leq m-1)$, *and furthermore the last row of* K_1^* *is expressed by* $-(m-s-t)\sqrt{2/(m-3)}(= w_1$, *say*$)$ *times the second one of* K_1^*,

(III) $\mathrm{rank}\{K_2^*\} \geq 1$, *and if* $\mathrm{rank}\{K_2^*\} = 1$, *then there exists* $\lambda_u \neq 0$ $(2 \leq u \leq m-2)$ *and* $\lambda_j = 0$ $(j \neq u;\ 2 \leq j \leq m-2)$, *and furthermore the last row of* K_2^* *is expressed by* $-(m-2u)/\sqrt{m-4}(= w_2$, *say*$)$ *times the first one of* K_2^*.

The proof will be given in Appendix C.

It follows from (2.2b) and (2.4) that (a) every element of $A_0^{\#(u,u)}\boldsymbol{\theta}_u$ $(0 \leq u \leq 3)$ represents the average of the u-factor interactions, (b) the elements of $A_\gamma^{\#(u,u)}\boldsymbol{\theta}_u$ $(1 \leq \gamma \leq u \leq 3)$ represent the contrasts among these effects, (c) any two contrasts $A_\beta^{\#(u,u)}\boldsymbol{\theta}_u$ and $A_\gamma^{\#(u,u)}\boldsymbol{\theta}_u$ $(1 \leq \beta \neq \gamma \leq u;\ 2 \leq u \leq 3)$ are orthogonal and (d) there exist ϕ_β independent parametric functions of $\boldsymbol{\theta}_u$ in $A_\beta^{\#(u,u)}\boldsymbol{\theta}_u$ $(0 \leq \beta \leq u \leq 3)$. Thus from (2.4), Proposition 2.3 and Lemmas 3.1 and 3.3, the following yields:

LEMMA 3.4. *Let T be a* 2^m-BFF *design of resolution* $\mathrm{R}^*(\{1\}|3)$ *with*

$\det(M_T)=0$, *and furthermore*

(I) *if* $\det(K_\beta^*)\neq 0$ *for some* $\beta\ (0\le\beta\le 3)$, *then*

$$A_\beta^{\#(\beta,\beta)}\boldsymbol{\theta}_\beta,\quad A_\beta^{\#(\beta+1,\beta+1)}\boldsymbol{\theta}_{\beta+1},\quad \ldots\quad ,\quad A_\beta^{\#(3,3)}\boldsymbol{\theta}_3$$

are estimable,

(II) *if* $\mathrm{rank}\{K_0^*\}=3$, *then*

$$\begin{aligned}
&g_0^{0,0}A_0^{\#(0,0)}\boldsymbol{\theta}_0+g_0^{0,3}A_0^{\#(0,3)}\boldsymbol{\theta}_3=g_0^{0,0}\left(A_0^{\#(0,0)}\boldsymbol{\theta}_0+w_0A_0^{\#(0,3)}\boldsymbol{\theta}_3\right),\\
&A_0^{\#(1,1)}\boldsymbol{\theta}_1,\\
&g_0^{2,2}A_0^{\#(2,2)}\boldsymbol{\theta}_2+g_0^{2,3}A_0^{\#(2,3)}\boldsymbol{\theta}_3=g_0^{2,2}\left(A_0^{\#(2,2)}\boldsymbol{\theta}_2+w_0^*A_0^{\#(2,3)}\boldsymbol{\theta}_3\right),\\
&g_0^{3,0}A_0^{\#(3,0)}\boldsymbol{\theta}_0+g_0^{3,2}A_0^{\#(3,2)}\boldsymbol{\theta}_2+g_0^{3,3}A_0^{\#(3,3)}\boldsymbol{\theta}_3\\
&=\left(u_0A_0^{\#(3,0)}\right)\left(g_0^{0,0}A_0^{\#(0,0)}\boldsymbol{\theta}_0+g_0^{0,3}A_0^{\#(0,3)}\boldsymbol{\theta}_3\right)\\
&\quad+\left(u_0^*A_0^{\#(3,2)}\right)\left(g_0^{2,2}A_0^{\#(2,2)}\boldsymbol{\theta}_2+g_0^{2,3}A_0^{\#(2,3)}\boldsymbol{\theta}_3\right)
\end{aligned}$$

are estimable, where w_0, w_0^*, u_0 *and* u_0^* *are the constants such that* $(\kappa_0^{3,1},\kappa_0^{3,0},\kappa_0^{3,2},\kappa_0^{3,3})=w_0(\kappa_0^{0,1},\kappa_0^{0,0},\kappa_0^{0,2},\kappa_0^{0,3})+w_0^*(\kappa_0^{2,1},\kappa_0^{2,0},\kappa_0^{2,2},\kappa_0^{2,3})$, $(g_0^{0,3},g_0^{2,3},g_0^{3,3})'=w_0(g_0^{0,0},0,g_0^{3,0})'+w_0^*(0,g_0^{2,2},g_0^{3,2})'$ *and* $(g_0^{3,0},g_0^{3,2},g_0^{3,3})=u_0(g_0^{0,0},0,g_0^{0,3})+u_0^*(0,g_0^{2,2},g_0^{2,3})$, *and* $g_0^{u,u}$ $(u=0,2)$ *are arbitrary,*

(III) *if* $\mathrm{rank}\{K_1^*\}=2$, *then*

$$\begin{aligned}
&A_1^{\#(1,1)}\boldsymbol{\theta}_1,\\
&g_1^{1,1}A_1^{\#(2,2)}\boldsymbol{\theta}_2+g_1^{1,2}A_1^{\#(2,3)}\boldsymbol{\theta}_3=g_1^{1,1}(A_1^{\#(2,2)}\boldsymbol{\theta}_2+w_1A_1^{\#(2,3)}\boldsymbol{\theta}_3),\\
&g_1^{2,1}A_1^{\#(3,2)}\boldsymbol{\theta}_2+g_1^{2,2}A_1^{\#(3,3)}\boldsymbol{\theta}_3=(u_1A_1^{\#(3,2)})(g_1^{1,1}A_1^{\#(2,2)}\boldsymbol{\theta}_2+g_1^{1,2}A_1^{\#(2,3)}\boldsymbol{\theta}_3)
\end{aligned}$$

are estimable, where w_1 *and* u_1 *are the constants such that* $(\kappa_1^{2,0},\kappa_1^{2,1},\kappa_1^{2,2})=w_1(\kappa_1^{1,0},\kappa_1^{1,1},\kappa_1^{1,2})$, $(g_1^{1,2},g_1^{2,2})'=w_1(g_1^{1,1},g_1^{2,1})'$ *and* $(g_1^{2,1},g_1^{2,2})=u_1(g_1^{1,1},g_1^{1,2})$, *and* $g_1^{1,1}$ *is arbitrary,*

(IV) *if* $\mathrm{rank}\{K_2^*\}=1$, *then*

$$\begin{aligned}
&g_2^{0,0}A_2^{\#(2,2)}\boldsymbol{\theta}_2+g_2^{0,1}A_2^{\#(2,3)}\boldsymbol{\theta}_3=g_2^{0,0}\left(A_2^{\#(2,2)}\boldsymbol{\theta}_2+w_2A_2^{\#(2,3)}\boldsymbol{\theta}_3\right),\\
&g_2^{1,0}A_2^{\#(3,2)}\boldsymbol{\theta}_2+g_2^{1,1}A_2^{\#(3,3)}\boldsymbol{\theta}_3=\left(u_2A_2^{\#(3,2)}\right)\left(g_2^{0,0}A_2^{\#(2,2)}\boldsymbol{\theta}_2+g_2^{0,1}A_2^{\#(2,3)}\boldsymbol{\theta}_3\right)
\end{aligned}$$

are estimable, where w_2 *and* u_2 *are the constants such that* $(\kappa_2^{1,0},\kappa_2^{1,1})=w_2(\kappa_2^{0,0},\kappa_2^{0,1})$, $(g_2^{0,1},g_2^{1,1})'=w_2(g_2^{0,0},g_2^{1,0})'$ *and* $(g_2^{1,0},g_2^{1,1})=u_2(g_2^{0,0},g_2^{0,1})$, *and* $g_2^{0,0}$ *is arbitrary.*

Note that from (2.4), we have $A_0^{\#(1,1)}+A_1^{\#(1,1)}=I_m$ and $A_0^{\#(1,1)}A_1^{\#(1,1)}=A_1^{\#(1,1)}A_0^{\#(1,1)}=O_{m\times m}$. Thus $\boldsymbol{\theta}_1$ is estimable if and only if $A_0^{\#(1,1)}\boldsymbol{\theta}_1$ and $A_1^{\#(1,1)}\boldsymbol{\theta}_1$ are estimable. The following can be easily proved:

LEMMA 3.5. (I) *Let x and y be integer variables of the homogeneous linear equation (HLE) or the homogeneous system of linear equations (HSLEs), where $0 \le x < y \le m$ and $m(\ge 6)$ is an integer, then we have the following :*

(i) *$m(m-2x)+(m-2x)(m-2y)+m(m-2y)+(3m-2) = 0$ and $(m-2x)(m-2y)+(m-2) = 0$ have $x = (k^2-k+2)/2$ and $y = (k^2+k+2)/2$, where $m = k^2+1\,(k \ge 3)$, and hence $m(m-2x)(m-2y)+m(3m-2x-2y) = 0$.*

(ii) *$(m-2x)(m-2y)+(m-2) = 0$ has* (a) *$y = n+1$ when $x = 1$, where $m = 2n+1\,(n \ge 3)$,* (b) *$x = n$ when $y = m-1$, where $m = 2n+1\,(n \ge 3)$,* (c) *$y = 4$ when $x = 2$, where $m = 6$,* (d) *$x = 2$ when $y = m-2$, where $m = 6$, and* (e) *$y = (m+1)/2+(x-1)/(m-2x)$ when $3 \le x < y \le m-3$, where y is an integer, $x < m/2 < y$ and $m-x-y \ne 0$.*

(iii) *$(m-2x)(m-2y)+(m-2) = 0$ and $m-x-y = 0$ have* (a) *$x = 2$ and $y = 4$ when $m = 6$, and* (b) *$x = (k^2-k+2)/2$ and $y = (k^2+k+2)/2$ when $m \ge 7$, and hence $3 \le x < y \le m-3$, where $m = k^2+2\,(k \ge 3)$.*

(II) *Let x, y and z be integer variables of the HLE or the HSLEs, where $0 \le x < y < z \le m$ and $m(\ge 6)$ is an integer, then we have the following :*

(i) *$(m-2x)(m-2y)+(m-2y)(m-2z)+(m-2z)(m-2x)+(3m-2) = 0$ has* (a) *$z = 5$ when $m = 6$, $x = 1$ and $y = 2$,* (b) *$z = 7$ when $m = 9$, $x = 1$ and $y = 2$,* (c) *no solution when $m \ne 6, 9$, $x = 1$ and $y = 2$,* (d) *$z = (3m-y+2)/4-y(y-5)/\{4(m-y-1)\}$ when $x = 1$, where z is an integer, $m \ge 7$ and $3 \le y < z \le m-3$, $(m-2x)(m-2y)(m-2z)+m(3m-2x-2y-2z) \ne 0$ and $3m-2x-2y-2z \ne 0$,* (e) *$y = 2(< 3)$ when $m = 9$, $x = 1$ and $z = m-2$,* (f) *no solution when $m \ne 9$, $x = 1$ and $z = m-2$,* (g) *$y = 3$ when $x = 1$ and $z = m-1$, where $m = 6$ and $3 \le y \le m-3$, and hence $(m-2x)(m-2y)(m-2z)+m(3m-2x-2y-2z) = 0$,* (h) *$z = (3m-y+1)/4-y(y-7)/\{4(m-y-2)\}$ when $x = 2$, where z is an integer, $m \ge 7$, $3 \le y < z \le m-3$, $(m-2x)(m-2y)(m-2z)+m(3m-2x-2y-2z) \ne 0$ and $3m-2x-2y-2z \ne 0$,* (i) *$y = 3, 4, 5$ or 6 when $x = 2$ and $z = m-2$, where $m = 9$, and hence $(m-2x)(m-2y)(m-2z)+m(3m-2x-2y-2z) \ne 0$ and $3m-2x-2y-2z \ne 0$,* (j) *$x = (k-1)(k-2)/6$, $y = (k+1)(k+2)/6$ and z is arbitrary when $3 \le x < y < z \le m-3$ and $m-x-y = 0$, where $m = (k^2+2)/3$, $y < z \le (k^2-7)/3$ and $k = 3h+1$ or $3h+2\,(h \ge 2)$, and hence $(m-2x)(m-2y)(m-2z)+m(3m-2x-2y-2z) \ne 0$ and $3m-2x-2y-2z \ne 0$, and* (k) *$z = m/2+\{(m-2x)(m-2y)+(3m-2)\}/\{4(m-x-y)\}$ when $3 \le x < y < z \le m-3$ and $m-x-y \ne 0$, where z is an integer, $m \ge 8$, $(m-2x)(m-2y)(m-2z)+m(3m-2x-2y-2z) \ne 0$ and $3m-2x-2y-2z \ne 0$.*

(ii) *$(m-2x)(m-2y)+(m-2y)(m-2z)+(m-2z)(m-2x)+(3m-2) = 0$ and $(m-2x)(m-2y)(m-2z)+m(3m-2x-2y-2z) = 0$ have* (a) *$y = (k^2-k+2)/2$ and $z = (k^2+k+2)/2$ when $x = 0$, where $m = k^2+1\,(k \ge 3)$, and hence $3m-2x-2y-2z \ne 0$, $(m-2y)(m-2z)+(m-2) = 0$ and $m-y-z \ne 0$,* (b) *$x = k(k-1)/2$ and $y = k(k+1)/2$ when $z = m$, where $m = k^2+1\,(k \ge 3)$, and hence $3m-2x-2y-2z \ne 0$, $(m-2x)(m-2y)+(m-2) = 0$ and $m-x-y \ne 0$,* (c) *$y = 3$ and $z = 5$ when $m = 6$ and $x = 1$, and* *hence* $3m-2x-2y-2z=0$, (d) $y=(m+1)/2+(2-f)/\{2(m-4)\}$ *and* $z=(m+1)/2+(2+f)/\{2(m-4)\}$ *when* $m \geq 7$ *and* $x=1$, *where* y *and* z *are integers and* $f=\sqrt{(m-2)^2+m(m-4)(m-6)}$ *is a positive integer, and hence* $3m-2x-2y-2z \neq 0$, (e) $x=1$ *and* $y=3$ *when* $m=6$ *and* $z=m-1$, *and hence* $3m-2x-2y-2z=0$, (f) $x=(m-1)/2-(2+f)/\{2(m-4)\}$ *and* $y=(m-1)/2-(2-f)/\{2(m-4)\}$ *when* $m \geq 7$ *and* $z=m-1$, *where* x *and* y *are integers, and* f *is the same equation as in* (d) *and it is an integer, and hence* $3m-2x-2y-2z \neq 0$, *and* (g) $y=\{m(m-2x)^2+(m-1)(m-2x)-m^2-f\}/[2\{(m-2x)^2-m\}]$ *and* $z=\{m(m-2x)^2+(m-1)(m-2x)-m^2+f\}/[2\{(m-2x)^2-m\}]$ *when* $2 \leq x < y < z \leq m-2$, *where* y *and* z *are integers,* $0 \leq x < (m-\sqrt{m})/2$, $3m-2x-2y-2z \neq 0$ *and* $f=\sqrt{m(m-2x)^4-(3m^2-1)(m-2x)^2+m^2(3m-2)}$ *is a positive integer.*

(iii) $(m-2x)(m-2y)+(m-2y)(m-2z)+(m-2z)(m-2x)+(3m-2)=0$ *and* $3m-2x-2y-2z=0$ *have* $y=(3m-2x-f)/4$ *and* $z=(3m-2x+f)/4$, *where* y *and* z *are integers and* $f=\sqrt{-3(m-2x)^2+12m-8}$ *is a positive integer. In particular,* (a) *if* $x=0$, *then* $-3(m-2x)^2+12m-8<0$ *for* $m \geq 6$, *and* (b) *if* $x=1$, *then* (b-1) $-3(m-2x)^2+12m-8>0$ *when* $m=6,7$, *and* (b-2) $-3(m-2x)^2+12m-8<0$ *when* $m \geq 8$. *When* $m=6$ *and* $x=1$, *we have* $f=4$, $y=3$ *and* $z=5$, *and hence* $(m-2x)(m-2y)(m-2z)=0$, *and when* $m=7$ *and* $x=1$, *we have* $f=1$, $y=9/2$ *and* $z=5$.

(iv) $(m-2x)(m-2y)+(m-2y)(m-2z)+(m-2z)(m-2x)+(3m-2)=0$, $(m-2x)(m-2y)(m-2z)+m(3m-2x-2y-2z)=0$ *and* $3m-2x-2y-2z=0$ *have* $x=(k-1)(k-2)/6$, $y=(k^2+2)/6$ *and* $z=(k+1)(k+2)/6$, *where* $m=(k^2+2)/3$ *and* $k=6h-2$ *or* $6h+2$ $(h \geq 1)$.

Remark 3.1. In Lemma 3.5, we have the following:

(A) The HLE given by (I)(ii)(e) has a solution (x,y). For example, $m=10$ and $(x,y)=(3,6)$.

(B) The HLEs given by (II)(i)(d), (h) and (k) have a solution (x,y,z). For example, $m=9$ and $(x,y,z)=(1,5,6)$ for (d), $m=14$ and $(x,y,z)=(2,7,9)$ for (h), and $m=13$ and $(x,y,z)=(3,7,9)$ for (k).

(C) In the HSLEs given by (II)(ii), the (g) case has a solution (x,y,z). For example, $m=21$ and $(x,y,z)=(6,10,14)$, where $f=240$. While the (d) and (f) cases have no solution for $7 \leq m \leq 30$.

(D) The HSLEs given by (II)(iii) has a solution (x,y,z). For example, $m=18$ and $(x,y,z)=(5,10,12)$, where $f=4$.

The following is the main theorem of this paper:

Theorem 3.1. *Let* T *be a* 2^m-BFF *design of resolution* $\mathrm{R}^*(\{1\}|3)$ *derived from an* $\mathrm{SA}(m;\{\lambda_i\})$, *where* $\det(M_T)=0$ *and* $m \geq 6$. *Then the following yields*:

(I) T *is of resolution* $\mathrm{R}(\{1\}|3)$ *if and only if one of* Table 3.1(i) *through* (vii) *holds,*

(II) *T is of resolution* R($\{0,1\}|3$) *if and only if one of* Table 3.2(i) *through* (vii) *holds,*

(III) *T is of resolution* R($\{1,2\}|3$), *i.e., resolution* VI, *if and only if* Table 3.3(i) *holds,*

(IV) *T is of resolution* R($\{0,1,2\}|3$), *i.e., resolution* VI, *if and only if one of* Table 3.4(i) *through* (v) *holds.*

The proof will be given in Appendix D.

Note that in Tables 3.1(i) through (iv), 3.2(i), (iv), (v) and (vi), and 3.4(i), an array given by (b) is the complementary array of (a) (e.g., Shirakura and Kuwada (1975)).

Table 3.1. Resolution R($\{1\}|3$) designs.

No.	indices of nonzero λ's	constraints	conditions
(i)(a)	1, 2, 5	6	
(b)	1, 4, 5	6	
(ii)(a)	1, 2, 7	9	
(b)	2, 7, 8	9	
(iii)(a)	$1,\ q,\ r$	m	$m \geq 7, \quad 3 \leq q < r \leq m-3$, $3m-2-2q-2r \neq 0$, $(m-2)(m-2q)(m-2r)$ $+m(3m-2-2q-2r) \neq 0$, $r = (3m-q+2)/4 - q(q-5)/\{4(m-q-1)\}$: integer
(b)	$p,\ q,\ m-1$	m	$m \geq 7, \quad 3 \leq p < q \leq m-3$, $m+2-2p-2q \neq 0$, $(m-2)(m-2p)(m-2q)$ $-m(m+2-2p-2q) \neq 0$, $q = 1+(m-1)(m-6)/\{4(p-1)\}$: integer
(iv)(a)	$2,\ q,\ r$	m	$m \geq 7, \quad 3 \leq q < r \leq m-3$, $3m-4-2q-2r \neq 0$, $(m-4)(m-2q)(m-2r)$ $+m(3m-4-2q-2r) \neq 0$, $r = (3m-q+1)/4 - q(q-7)/\{4(m-q-2)\}$: integer
(b)	$p,\ q,\ m-2$	m	$m \geq 7, \quad 3 \leq p < q \leq m-3$, $m+4-2p-2q \neq 0$, $(m-4)(m-2p)(m-2q)$ $-m(m+4-2p-2q) \neq 0$, $q = 2+(m-2)(m-9)/\{4(p-2)\}$: integer
(v)	$2,\ q,\ 7$	9	$3 \leq q \leq 6$
(vi)	$(k-1)(k-2)/6$, $(k+1)(k+2)/6$, r	$(k^2+2)/3$	$k = 3h+1$ or $3h+2$ $(h \geq 2)$, $(k+1)(k+2)/6 < r \leq (k^2-7)/3$
(vii)	$p,\ q,\ r$	m	$m \geq 8, \quad 3 \leq p < q < r \leq m-3$, $(m-2p)(m-2q)+(m-2q)(m-2r)$ $+(m-2r)(m-2p)+(3m-2) = 0$, $(m-2p)(m-2q)(m-2r)$ $+m(3m-2p-2q-2r) \neq 0$, $3m-2p-2q-2r \neq 0$

Table 3.2. Resolution R($\{0,1\}|3$) designs.

No.	indices of nonzero λ's	constraints	conditions
(i)(a)	0 (or m), $1,\ 2,\ m-1$	m	
(b)	0 (or m), $1,\ m-2,\ m-1$	m	
(ii)	0 (or m), $1,\ u,\ m-1$	m	$m \geq 7, \quad 3 \leq u\ (\neq m/2) \leq m-3$
(iii)	$0,\ s,\ t,\ m$	m	$m \geq 7, \quad 3 \leq s < m/2 < t \leq m-3$, $m-s-t \neq 0$, $t = (m+1)/2 + (s-1)/(m-2s)$: integer
(iv)(a)	$0,\ 1,\ n+1,\ 2n+1$	$2n+1$	$n \geq 3$
(b)	$0,\ n,\ 2n,\ 2n+1$	$2n+1$	$n \geq 3$
(v)(a)	$0,\ (k^2-k+2)/2$, $(k^2+k+2)/2$	k^2+1	$k \geq 3$
(b)	$k(k-1)/2,\ k(k+1)/2$, k^2+1	k^2+1	$k \geq 3$
(vi)(a)	$1,\ q,\ r$	m	$m \geq 7, \quad 3 \leq q < r \leq m-3$, $q = (m+1)/2 + (2-f)/\{2(m-4)\}$: integer, $r = (m+1)/2 + (2+f)/\{2(m-4)\}$: integer, $f = \{(m-2)^2 + m(m-4)(m-6)\}^{1/2}$: positive integer
(b)	$p,\ q,\ m-1$	m	$m \geq 7, \quad 3 \leq p < q \leq m-3$, $p = (m-1)/2 - (2+f)/\{2(m-4)\}$: integer, $q = (m-1)/2 - (2-f)/\{2(m-4)\}$: integer, $f = \{(m-2)^2 + m(m-4)(m-6)\}^{1/2}$: positive integer
(vii)	$p,\ q,\ r$	m	$2 \leq p < q < r \leq m-2, \quad \sqrt{m} < m-2p \leq m$, $3m-2p-2q-2r \neq 0$, $q = \{m(m-2p)^2 + (m-1)(m-2p) - m^2 - f\}$ $/[2\{(m-2p)^2 - m\}]$: integer, $r = \{m(m-2p)^2 + (m-1)(m-2p) - m^2 + f\}$ $/[2\{(m-2p)^2 - m\}]$: integer, $f = \{m(m-2p)^4$ $-(3m^2-1)(m-2p)^2 + m^2(3m-2)\}^{1/2}$: positive integer

Table 3.3. Resolution R($\{1,2\}|3$) designs.

No.	indices of nonzero λ's	constraints	conditions
(i)	$p,\ q,\ r$	m	$2 \leq p < q < r \leq m-2, \quad (m-2p)(m-2q)(m-2r) \neq 0$, $q = (3m-2p-f)/4$: integer, $r = (3m-2p+f)/4$: integer, $f = \{-3(m-2p)^2 + 12m - 8\}^{1/2}$: positive integer

Table 3.4. Resolution R($\{0,1,2\}|3$) designs.

No.	indices of nonzero λ's	constraints	conditions
(i)(a)	0 (or m), $1,\ 2,\ m-2$	m	
(b)	0 (or m), $2,\ m-2,\ m-1$	m	
(ii)	$1,\ 2,\ m-2,\ m-1$	m	$\lambda_0 \geq 0, \quad \lambda_m \geq 0$
(iii)	0 (or $2n$), $1,\ n,\ 2n-1$	$2n$	$n \geq 3$
(iv)	$0,\ (k^2-k+2)/2,\ (k^2+k+2)/2,\ k^2+2$	k^2+2	$k \geq 2$
(v)	$(k-1)(k-2)/6,\ (k^2+2)/6,\ (k+1)(k+2)/6$	$(k^2+2)/3$	$k = 6h-2$ or $6h+2$ $(h \geq 1)$

4. GA-optimal designs

If $N \geq \nu_3$, then there exists a 2^m-BFF design of resolution VII (e.g., Shirakura (1976)). Thus in this section, we only consider a design with $N < \nu_3$, and hence $\det(M_T) = 0$, i.e., $\det(K^*_\beta) = 0$ for some β $(0 \le \beta \le 3)$. Since $2\binom{m}{3} - \nu_3 = \{(m^2+m+6)(m-7)+36\}/6 > 0$ for $m \ge 7$, and $2\binom{m}{2} + \binom{m}{3} - \nu_3 = \{m(m-6)+3(m-6)+16\}/2 > 0$ for $m \ge 6$, we have the following:

LEMMA 4.1. *$2\binom{m}{3} > \nu_3$ for $m \ge 7$, and $2\binom{m}{2} + \binom{m}{3} - \nu_3 > 0$ for $m \ge 6$.*

Remark 4.1. It follows from Lemma 4.1 that 2^m-BFF designs of resolution $\mathrm{R}^*(\{1\}|3)$ given by Tables 3.1(iii) through (vii), 3.2(iii), (v), (vi), (vii), 3.3(i), and 3.4(iv) except for $k = 2$, and (v) except for $k = 4$ have $N \ge \nu_3$, and the remaining have $N < \nu_3$.

As shown in Section 3, if a parametric function $C\boldsymbol{\Theta}$ is estimable (and hence there exists a matrix X such that $XM_T = C$), then $\mathrm{Var}[C\hat{\boldsymbol{\Theta}}] = \sigma^2 XM_TX'$. Using a solution Z of the matrix equation $ZL = H$ given by Lemma 3.1, after some calculations, we have

$$ZLZ' = \begin{cases} L_{11}^{-1} & \text{if } n_2 = n_3 = 0, \\ \begin{pmatrix} I_{n_1} \\ 0 \end{pmatrix} L_{11}^{-1}\,(I_{n_1};\ 0) & \text{if } n_2 = 0 \text{ and } n_3 \ge 1, \\ \begin{pmatrix} I_{n_1} & 0 \\ 0 & H_{22} \end{pmatrix} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}^{-1} \begin{pmatrix} I_{n_1} & 0 \\ 0 & H'_{22} \end{pmatrix} & \text{if } n_2 \ge 1 \text{ and } n_3 = 0, \\ \begin{pmatrix} I_{n_1} & 0 \\ 0 & H_{22} \\ 0 & H_{32} \end{pmatrix} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}^{-1} \begin{pmatrix} I_{n_1} & 0 & 0 \\ 0 & H'_{22} & H'_{32} \end{pmatrix} & \text{if } n_2 \ge 1 \text{ and } n_3 \ge 1, \end{cases}$$

where H_{22} and H_{32} are arbitrary. Since C and X belong to the TMDPB association algebra $\mathcal{A}$, XM_TX' is isomorphic to $\chi_\beta K_\beta \chi'_\beta$ for $0 \le \beta \le 3$. Thus we can get

$$\text{(I)}\qquad \chi^*_0 K^*_0 \chi^{*\prime}_0 = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & g_0^{0,0} & 0 \\ 0 & 0 & g_0^{2,2} \\ 0 & g_0^{3,0} & g_0^{3,2} \end{pmatrix} \begin{pmatrix} \kappa_0^{1,1} & \kappa_0^{1,0} & \kappa_0^{1,2} \\ \kappa_0^{0,1} & \kappa_0^{0,0} & \kappa_0^{0,2} \\ \kappa_0^{2,1} & \kappa_0^{2,0} & \kappa_0^{2,2} \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & g_0^{0,0} & 0 & g_0^{3,0} \\ 0 & 0 & g_0^{2,2} & g_0^{3,2} \end{pmatrix}, & \\ & \text{if } \mathrm{rank}\{K^*_0\} = 3, \\ K_0^{*-1} & \text{if } \mathrm{rank}\{K^*_0\} = 4, \end{cases}$$

$$\text{(II)}\qquad \chi^*_1 K^*_1 \chi^{*\prime}_1 = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & g_1^{1,1} \\ 0 & g_1^{2,1} \end{pmatrix} \begin{pmatrix} \kappa_1^{0,0} & \kappa_1^{0,1} \\ \kappa_1^{1,0} & \kappa_1^{1,1} \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & g_1^{1,1} & g_1^{2,1} \end{pmatrix} & \text{if } \mathrm{rank}\{K^*_1\} = 2, \\ K_1^{*-1} & \text{if } \mathrm{rank}\{K^*_1\} = 3, \end{cases}$$

$$\text{(III)}\qquad \chi^*_2 K^*_2 \chi^{*\prime}_2 = \begin{cases} \begin{pmatrix} g_2^{0,0} \\ g_2^{1,0} \end{pmatrix} \begin{pmatrix} \kappa_2^{0,0} \end{pmatrix}^{-1} \begin{pmatrix} g_2^{0,0} & g_2^{1,0} \end{pmatrix} & \text{if } \mathrm{rank}\{K^*_2\} = 1, \\ K_2^{*-1} & \text{if } \mathrm{rank}\{K^*_2\} = 2, \end{cases}$$

$$\text{(IV)}\quad \chi_3^* K_3^* \chi_3^{*\prime} = \begin{cases} \text{vanish} & \text{if } \operatorname{rank}\{K_3^*\} = 0, \\ K_3^{*-1} & \text{if } \operatorname{rank}\{K_3^*\} = 1, \end{cases}$$

where χ_β^* and Γ_β^* are given in Section 3, and $g_0^{u,u}$ $(u = 0, 2)$ and $g_\gamma^{2-\gamma,2-\gamma}$ $(1 \leq \gamma \leq 2)$ are arbitrary, and furthermore there exist constants u_0, u_0^* and u_γ such that $g_0^{3,0} = u_0 g_0^{0,0}$, $g_0^{3,2} = u_0^* g_0^{2,2}$ and $g_\gamma^{3-\gamma,2-\gamma} = u_\gamma g_\gamma^{2-\gamma,2-\gamma}$, respectively. Thus from Lemma 3.4, if $\operatorname{rank}\{K_0^*\} = 3$, then we put

$$g_0^{0,0} \left(= g_0^{0,0}(\alpha),\ \text{say}\right) = \begin{cases} 1 & \text{if } \alpha = 0, \\ 1/(1+|w_0|) & \text{if } \alpha = 1, \\ 1\Big/\sqrt{1+(w_0)^2} & \text{if } \alpha = 2, \end{cases}$$

$$g_0^{2,2} \left(= g_0^{2,2}(\alpha),\ \text{say}\right) = \begin{cases} 1 & \text{if } \alpha = 0, \\ 1/(1+|w_0^*|) & \text{if } \alpha = 1, \\ 1\Big/\sqrt{1+(w_0^*)^2} & \text{if } \alpha = 2, \end{cases}$$

and if $\operatorname{rank}\{K_\gamma^*\} = 3 - \gamma$ $(1 \leq \gamma \leq 2)$, then we put

$$g_\gamma^{2-\gamma,2-\gamma} \left(= g_\gamma^{2-\gamma,2-\gamma}(\alpha),\ \text{say}\right) = \begin{cases} 1 & \text{if } \alpha = 0, \\ 1/(1+|w_\gamma|) & \text{if } \alpha = 1, \\ 1\Big/\sqrt{1+(w_\gamma)^2} & \text{if } \alpha = 2, \end{cases}$$

where (I) if $\operatorname{rank}\{K_0^*\} = 3$, then the last row of K_0^* is expressed by the sum of w_0 times the second one of K_0^* and of w_0^* times the third, (II) if $\operatorname{rank}\{K_1^*\} = 2$, then the last row of K_1^* is expressed by w_1 times the second one of K_1^*, and (III) if $\operatorname{rank}\{K_2^*\} = 1$, then the last row of K_2^* is expressed by w_2 times the first one of K_2^*. Here w_0, w_0^*, w_1 and w_2 are given in Lemma 3.3.

Let

$$\tilde{\chi}_0^*(\alpha) = \begin{cases} \operatorname{diag}[1; g_0^{0,0}(\alpha); g_0^{2,2}(\alpha)] & \text{if } \operatorname{rank}\{K_0^*\} = 3, \\ I_4 & \text{if } \operatorname{rank}\{K_0^*\} = 4, \end{cases}$$

$$\tilde{\chi}_1^*(\alpha) = \begin{cases} \operatorname{diag}[1; g_1^{1,1}(\alpha)] & \text{if } \operatorname{rank}\{K_1^*\} = 2, \\ I_3 & \text{if } \operatorname{rank}\{K_1^*\} = 3, \end{cases}$$

$$\tilde{\chi}_2^*(\alpha) = \begin{cases} g_2^{0,0}(\alpha) & \text{if } \operatorname{rank}\{K_2^*\} = 1, \\ I_2 & \text{if } \operatorname{rank}\{K_2^*\} = 2, \end{cases}$$

$$\tilde{\chi}_3^*(\alpha) = \begin{cases} \text{vanish} & \text{if } \operatorname{rank}\{K_3^*\} = 0, \\ 1 & \text{if } \operatorname{rank}\{K_3^*\} = 1. \end{cases}$$

Further let $\tilde{K}_\beta^*$ be the matrices given by the first r_β rows and columns of K_β^*, where $r_\beta = \operatorname{rank}\{K_\beta^*\} \geq 1$ for $0 \leq \beta \leq 3$. Then from Proposition 2.1 and Lemma 3.4, the variance-covariance matrix of the linearly independent estimators in $C\hat{\boldsymbol{\Theta}}$

is isomorphic to $\sigma^2 \tilde{\chi}^*_\beta(\alpha)\tilde{K}^{*-1}_\beta \tilde{\chi}^*_\beta(\alpha)'$ for $0 \le \beta \le 3$ $(0 \le \alpha \le 2)$. Thus for a 2^m-BFF design T of resolution $\mathrm{R}^*(\{1\}|3)$ derived from an $\mathrm{SA}(m;\{\lambda_i\})$, we define $S_T(\alpha)$ as follows:

$$S_T(\alpha) = \sum_\beta \phi_\beta \operatorname{tr}\{\tilde{\chi}^*_\beta(\alpha)\tilde{K}^{*-1}_\beta \tilde{\chi}^*_\beta(\alpha)'\},$$

where $\sum_\beta$ is the summation over all the values of β such that if $\mathrm{rank}\{K^*_3\} = 0$, then $0 \le \beta \le 2$, and if $\mathrm{rank}\{K^*_3\} = 1$, then $0 \le \beta \le 3$, and ϕ_β is given by (2.3). Note that $\sigma^2 S_T(\alpha)$ are the trace of the variance-covariance matrix of the linearly independent estimators in $C\hat{\boldsymbol{\Theta}}$, and hence the GA-optimality criterion that will be defined below is based on the average of the variances of the linearly independent estimators. Thus in a sense, it refers to the average variance. The following is due to Kuwada *et al.* (2002):

Definition 4.1. *Let T be a 2^m-BFF design of resolution* $\mathrm{R}^*(\{1\}|3)$ *with N assemblies derived from an* $\mathrm{SA}(m;\{\lambda_i\})$. *If $S_T(\alpha) \le S_{T^*}(\alpha)$ for any T^* being a 2^m-BFF design of resolution* $\mathrm{R}^*(\{1\}|3)$ *with N assemblies derived from an* $\mathrm{SA}(m;\{\lambda^*_i\})$, *then T is said to be* GA_α*-optimal* $(0 \le \alpha \le 2)$.

Using Theorem 3.1 and Remark 4.1, we can obtain GA_α-optimal 2^m-BFF designs of resolution $\mathrm{R}^*(\{1\}|3)$, where $N < \nu_3$. All GA_α-optimal designs for $6 \le m \le 8$ are the same designs as GA_α-optimal ones of resolution $\mathrm{R}^*(\{0,1\}|3)$ (see Kuwada *et al.* (2002)) except for $m = 6$ and $(N,\alpha) = (27,0),\ (27,1),\ (27,2),\ (39,1)$. While GA_α-optimal 2^6-BFF designs of resolution $\mathrm{R}^*(\{1\}|3)$ with $N = 27$ and α $(0 \le \alpha \le 2)$ and with $N = 39$ and $\alpha = 1$ are given by $\mathrm{SA}(6;\{0,1,0,0,1,1,0\})$ and its complement and $\mathrm{SA}(6;\{0,1,0,0,1,3,0\})$ and its complement, respectively, where $\{\lambda_i\} = \{\lambda_0,\lambda_1,\dots,\lambda_6\}$. Note that both designs are given by Table 3.1(i), and that we have $S_T(0) = 1.5353,\ S_T(1) = 0.9844$ and $S_T(2) = 1.1200$ for $N = 27$, and $S_T(1) = 0.7359$ for $N = 39$.

Appendix A: Relationship between $\kappa^{u,v}_\beta$'s and λ_i's

$$\kappa^{0,0}_0 (= N) = \sum_{i=0}^m \binom{m}{i} \lambda_i,$$

$$\kappa^{0,1}_0 \left(= \kappa^{1,0}_0\right) = -\left(1 \big/ \sqrt{m}\right) \sum_{i=0}^m \binom{m}{i} (m-2i)\lambda_i,$$

$$\kappa^{0,2}_0 \left(= \kappa^{2,0}_0\right) = \left[1 \Bigg/ \left\{2\sqrt{\binom{m}{2}}\right\}\right] \sum_{i=0}^m \binom{m}{i} \left\{(m-2i)^2 - m\right\} \lambda_i,$$

$$\kappa_0^{0,3}\left(=\kappa_0^{3,0}\right) = -\left[1\left/\left\{6\sqrt{\binom{m}{3}}\right\}\right.\right] \times \sum_{i=0}^{m}\binom{m}{i}(m-2i)\left\{(m-2i)^2-(3m-2)\right\}\lambda_i,$$

$$\kappa_0^{1,1} = (1/m)\sum_{i=0}^{m}\binom{m}{i}(m-2i)^2\lambda_i,$$

$$\kappa_0^{1,2}\left(=\kappa_0^{2,1}\right) = -\left[1\left/\left\{m\sqrt{2(m-1)}\right\}\right.\right]\sum_{i=0}^{m}\binom{m}{i}(m-2i)\left\{(m-2i)^2-m\right\}\lambda_i,$$

$$\kappa_0^{1,3}\left(=\kappa_0^{3,1}\right) = \left[1\left/\left\{2m\sqrt{3\binom{m-1}{2}}\right\}\right.\right] \times \sum_{i=0}^{m}\binom{m}{i}(m-2i)^2\left\{(m-2i)^2-(3m-2)\right\}\lambda_i,$$

$$\kappa_0^{2,2} = \left[1\left/\left\{4\binom{m}{2}\right\}\right.\right]\sum_{i=0}^{m}\binom{m}{i}\left\{(m-2i)^2-m\right\}^2\lambda_i,$$

$$\kappa_0^{2,3}\left(=\kappa_0^{3,2}\right) = -\left[1\left/\left\{4\binom{m}{2}\sqrt{3(m-2)}\right\}\right.\right] \times \sum_{i=0}^{m}\binom{m}{i}(m-2i)\left\{(m-2i)^2-m\right\}\left\{(m-2i)^2-(3m-2)\right\}\lambda_i,$$

$$\kappa_0^{3,3} = \left[1\left/\left\{36\binom{m}{3}\right\}\right.\right]\sum_{i=0}^{m}\binom{m}{i}(m-2i)^2\left\{(m-2i)^2-(3m-2)\right\}^2\lambda_i,$$

$$\kappa_1^{0,0} = 4\sum_{j=1}^{m-1}\binom{m-2}{j-1}\lambda_j,$$

$$\kappa_1^{0,1}\left(=\kappa_1^{1,0}\right) = -\left(4\left/\sqrt{m-2}\right.\right)\sum_{j=1}^{m-1}\binom{m-2}{j-1}(m-2j)\lambda_j,$$

$$\kappa_1^{0,2}\left(=\kappa_1^{2,0}\right) = \left\{2\left/\sqrt{\binom{m-2}{2}}\right.\right\}\sum_{j=1}^{m-1}\binom{m-2}{j-1}\left\{(m-2j)^2-(m-2)\right\}\lambda_j,$$

$$\kappa_1^{1,1} = \{4/(m-2)\}\sum_{j=1}^{m-1}\binom{m-2}{j-1}(m-2j)^2\lambda_j,$$

$$\kappa_1^{1,2}\left(=\kappa_1^{2,1}\right) = -\left[4\left/\left\{(m-2)\sqrt{2(m-3)}\right\}\right.\right] \times \sum_{j=1}^{m-1}\binom{m-2}{j-1}(m-2j)\left\{(m-2j)^2-(m-2)\right\}\lambda_j,$$

$$\kappa_1^{2,2} = \left\{1\left/\binom{m-2}{2}\right.\right\}\sum_{j=1}^{m-1}\binom{m-2}{j-1}\left\{(m-2j)^2-(m-2)\right\}^2\lambda_j,$$

$$\kappa_2^{0,0} = 16 \sum_{k=2}^{m-2} \binom{m-4}{k-2} \lambda_k,$$

$$\kappa_2^{0,1} \left(= \kappa_2^{1,0}\right) = -\left(16/\sqrt{m-4}\right) \sum_{k=2}^{m-2} \binom{m-4}{k-2} (m-2k)\lambda_k,$$

$$\kappa_2^{1,1} = \{16/(m-4)\} \sum_{k=2}^{m-2} \binom{m-4}{k-2} (m-2k)^2 \lambda_k,$$

$$\kappa_3^{0,0} = 64 \sum_{h=3}^{m-3} \binom{m-6}{h-3} \lambda_h.$$

Appendix B: The proof of Lemma 3.2

It follows from Remark 2.1 that the first, the second, the third and the last rows (and columns) of K_0^* correspond to the main effects, the general mean, the two-factor interactions and the three-factor ones, respectively, and that the first, the second and the last rows (and columns) of K_1^* correspond to the main effects, the two-factor ones and the three-factor ones, respectively. Thus from Proposition 2.2 and Appendix A, if rank$\{K_1^*\} = 1$, then the second and the last rows of K_1^* are expressed by $-(m-2p)/\sqrt{m-2}$ times the first one of K_1^*, and by $\{(m-2p)^2 - (m-2)\}/\sqrt{2(m-2)(m-3)}$ times the first, respectively, where $\lambda_p \neq 0$ $(1 \leq p \leq m-1)$ and $\lambda_j = 0$ $(j \neq p; 1 \leq j \leq m-1)$. Hence from Lemma 3.1, if the main effects are estimable, then we have $m - 2p = 0$ and $(m-2p)^2 - (m-2) = 0$. However there does not exist an integer p such that $m - 2p = 0$ and $(m-2p)^2 - (m-2) = 0$ for $m \geq 6$. Thus at least two of λ_i $(1 \leq i \leq m-1)$ are nonzero, and hence rank$\{K_1^*\} \geq 2$ and rank$\{K_0^*\} \geq 2$. On the other hand, if rank$\{K_0^*\} = 2$, then exactly two of λ_i $(0 \leq i \leq m)$ are nonzero, i.e., $\lambda_q \neq 0$ and $\lambda_r \neq 0$ $(0 \leq q < r \leq m)$. In this case, the third and the last rows of K_0^* are expressed by the sum of $-(m-q-r)\sqrt{2/(m-1)}$ times the first one of K_0^* and of $-\{(m-2q)(m-2r)+m\}/\sqrt{2m(m-1)}$ times the second, and by the sum of $\{4(m-q-r)^2 - (m-2q)(m-2r) - (3m-2)\}/\sqrt{6(m-1)(m-2)}$ times the first and of $\sqrt{2}(m-2q)(m-2r)(m-q-r)/\sqrt{3m(m-1)(m-2)}$ times the second, respectively. Furthermore from the results mentioned above, $1 \leq q < r \leq m-1$ holds, and hence rank$\{K_1^*\} = 2$. Thus the last row of K_1^* is expressed by the sum of $-\{(m-2q)(m-2r)+(m-2)\}/\sqrt{2(m-2)(m-3)}$ times the first one of K_1^* and of $-\sqrt{2}(m-q-r)/\sqrt{m-3}$ times the second. Hence from Lemma 3.1, if the main effects are estimable, then $m-q-r = 0$, $4(m-q-r)^2 - (m-2q)(m-2r) - (3m-2) = 0$ and $(m-2q)(m-2r) + (m-2) = 0$ hold. However there do not exist integers q and r $(1 \leq q < r \leq m-1)$ such that $m-q-r = 0$, $4(m-q-r)^2 - (m-2q)(m-2r) - (3m-2) = 0$ and $(m-2q)(m-2r) + (m-2) = 0$ for $m \geq 6$. Thus at least three of λ_i $(0 \leq i \leq m)$ are nonzero, and hence rank$\{K_0^*\} \geq 3$. Therefore we have the required results.

Appendix C: The proof of Lemma 3.3

From Proposition 2.2, Lemma 3.2 and Appendix A, if rank$\{K_0^*\} = 3$, then the last row of K_0^* is expressed by the sum of $-\{(m-2p)(m-2q)+(m-2q)(m-2r)+(m-2r)(m-2p)+(3m-2)\}/\sqrt{6(m-1)(m-2)}$ times the first one of K_0^*, of w_0 times the second and of w_0^* times the third, where $\lambda_i \neq 0$ $(i = p, q, r; 0 \leq p < q < r \leq m)$ and $\lambda_j = 0$ $(j \neq p, q, r; 0 \leq j \leq m)$, and if rank$\{K_1^*\} = 2$, then the last row of K_1^* is expressed by the sum of $-\{(m-2s)(m-2t)+(m-2)\}/\sqrt{2(m-2)(m-3)}$ times the first one of K_1^* and of w_1 times the second, where $\lambda_i \neq 0$ $(i = s, t; 1 \leq s < t \leq m-1)$ and $\lambda_j = 0$ $(j \neq s, t; 1 \leq j \leq m-1)$. Thus from Lemma 3.1, if T is of resolution R$^*(\{1\}|3)$ with $\det(M_T) = 0$, then $\tilde{w}_0 = \tilde{w}_1 = 0$ hold. Moreover from Proposition 2.2, rank$\{K_2^*\} = 0$ if and only if $\lambda_i = 0$ for all i $(2 \leq i \leq m-2)$, and hence we have rank$\{K_1^*\} \leq 2$. On the other hand, from Lemma 3.2, we have rank$\{K_1^*\} \geq 2$, and hence rank$\{K_1^*\} = 2$, i.e., $\lambda_1 \neq 0$ and $\lambda_{m-1} \neq 0$. However from (II), the suffixes of λ_1 and λ_{m-1}, i.e., $s = 1$ and $t = m-1$, do not satisfy the condition such that $\tilde{w}_1 = 0$ for $m \geq 6$, and hence rank$\{K_2^*\} \geq 1$. Furthermore from Remark 2.1, Proposition 2.2 and Appendix A, if rank$\{K_2^*\} = 1$, then the last row of K_2^* is expressed by w_2 times the first one of K_2^*, where $\lambda_u \neq 0$ $(2 \leq u \leq m-2)$ and $\lambda_j = 0$ $(j \neq u; 2 \leq j \leq m-2)$. Therefore the proof is complete.

Appendix D: The proof of Theorem 3.1

If rank$\{K_0^*\} = 4$, then from Lemma 3.4, the general mean is estimable. Thus if the general mean is confounded (or aliased) with the remaining effects, then rank$\{K_0^*\} = 3$ and $w_0 \neq 0$ hold, where $\lambda_i \neq 0$ $(i = p, q, r; 0 \leq p < q < r \leq m)$, $\lambda_j = 0$ $(j \neq p, q, r; 0 \leq j \leq m)$ and w_0 is given in Lemma 3.3.

(I) Let T be of resolution R$(\{1\}|3)$, then from the results mentioned above, we have rank$\{K_0^*\} = 3$, and hence from Lemma 3.3, there exist $\lambda_i \neq 0$ $(i = p, q, r; 0 \leq p < q < r \leq m)$ such that $\tilde{w}_0 = 0$, and $\lambda_j = 0$ $(j \neq p, q, r; 0 \leq j \leq m)$, where $\tilde{w}_0$ is given in Lemma 3.3. Moreover (i) if $p = 0$, then from Lemma 3.2, we have rank$\{K_1^*\} = 2$, i.e., $1 \leq q < r \leq m-1$, and hence from Lemma 3.3, $\tilde{w}_1 = 0$ holds, where put $s = q$ and $t = r$ in $\tilde{w}_1$ given in Lemma 3.3. However from Lemma 3.5(I)(i), we have $w_0 = 0$, and hence there does not exist an SA$(m; \{\lambda_i\})$. (ii) If $p = 1$, and furthermore (ii-a) if $q = 2 < r \leq m$, then from Lemma 3.5(II)(i)(a), (b) and (c), we get $m = 6$ and $r = 5$, and $m = 9$ and $r = 7$. When $m = 6$, $p = 1$, $q = 2$ and $r = 5$, we have $w_0 \neq 0$, $w_0^* \neq 0$, $w_2 \neq 0$, rank$\{K_1^*\} = 3$, rank$\{K_2^*\} = 1$ and $K_3^* = 0$, where w_0^* is given in Lemma 3.3 and put $u = q$ in w_2 given in Lemma 3.3, and when $m = 9$, $p = 1$, $q = 2$ and $r = 7$, we have $w_0 \neq 0$, $w_0^* \neq 0$, rank$\{K_\gamma^*\} = 4 - \gamma$ $(1 \leq \gamma \leq 2)$ and $K_3^* = 0$. Thus we get Table 3.1(i)(a) and (ii)(a). (ii-b) If $3 \leq q < r \leq m-3$ and $m \geq 7$, then from Lemma 3.5(II)(i)(d), we have rank$\{K_\gamma^*\} = 4 - \gamma$ $(1 \leq \gamma \leq 3)$, where $w_0 \neq 0$ and $w_0^* \neq 0$, and hence Table 3.1(iii)(a), (ii-c) if $3 \leq q < r = m-2$, then from Lemma 3.5(II)(i)(e) and (f), there does not exist an array, and (ii-d) if $3 \leq q \leq m-3$ and $r = m-1$, then from Lemma 3.5(II)(i)(g), we get $m = 6$ and $q = 3$, and hence there does not exist an array since $w_0 = 0$. (iii) If $p = 2$, and furthermore (iii-a) if $3 \leq q < r \leq m-3$, then from Lemma

3.5(II)(i)(h), we have rank$\{K^*_\gamma\} = 4 - \gamma$ $(1 \le \gamma \le 3)$, where $w_0 \ne 0$ and $w^*_0 \ne 0$, and hence Table 3.1(iv)(a), and (iii-b) if $3 \le q \le m-3$ and $r = m-2$, then from Lemma 3.5(II)(i)(i), we get $m = 9$, $3 \le q \le 6$, $w_0 \ne 0$, $w^*_0 \ne 0$ and rank$\{K^*_\gamma\} = 4 - \gamma$ $(1 \le \gamma \le 3)$, and hence we get Table 3.1(v). (iv) If $3 \le p < q < r \le m-3$, $m-p-q = 0$ and $m \ge 8$, then from Lemma 3.5(II)(i)(j), we have $w_0 \ne 0$, $w^*_0 \ne 0$ and rank$\{K^*_\gamma\} = 4 - \gamma$ $(1 \le \gamma \le 3)$, and hence Table 3.1(vi). (v) If $3 \le p < q < r \le m-3$, $m-p-q \ne 0$ and $m \ge 8$, then from Lemma 3.5(II)(i)(k), we have rank$\{K^*_\gamma\} = 4 - \gamma$ $(1 \le \gamma \le 3)$, where $w_0 \ne 0$ and $w^*_0 \ne 0$, and hence Table 3.1(vii). (vi) If $1 \le p < q < r = m$ or $p = 2 < q \le m-3$ and $r = m-1$, then from Proposition 2.4 and in addition (i) or (ii-c) mentioned above, there does not exist an array. (vii) By using Proposition 2.4, Table 3.1(i)(b), (ii)(b), (iii)(b) and (iv)(b) can be easily obtained. Conversely if one of Table 3.1(i) through (vii) holds, then from Lemmas 3.1 and 3.3, it can be easily shown that T is of resolution R$(\{1\}|3)$.

(II) Let T be of resolution R$(\{0,1\}|3)$. Then (A) if rank$\{K^*_0\} = 4$ and in addition, (i) if rank$\{K^*_1\} = 3$, and furthermore if rank$\{K^*_2\} = 2$, then from Lemma 3.4, all the factorial effects up to the two-factor interactions are estimable. Thus rank$\{K^*_2\} = 1$, i.e., $\lambda_0 + \lambda_m \ne 0$, $\lambda_i \ne 0$ $(i = 1, u, m-1; 2 \le u \le m-2)$ and $\lambda_j = 0$ $(j \ne 0, 1, u, m-1, m; 2 \le j \le m-2)$, and hence from Lemma 3.4, $w_2 \ne 0$ holds. In particular, (i-a) if $u = 2$ or $m-2$, then we have $w_2 \ne 0$ and $K^*_3 = 0$, and hence we get Table 3.2(i)(a) and (b), and (i-b) if $m \ge 7$ and $3 \le u \ne m/2 \le m-3$, then we have $w_2 \ne 0$ and $K^*_3 \ne 0$, and hence Table 3.2(ii). (ii) If rank$\{K^*_1\} = 2$, i.e., $\lambda_i \ne 0$ $(i = 0, s, t, m; 1 \le s < t \le m-1)$ and $\lambda_j = 0$ $(j \ne 0, s, t, m; 1 \le j \le m-1)$, then from Lemma 3.3, $\tilde{w}_1 = 0$ holds, and furthermore (ii-a) if rank$\{K^*_2\} = 2$, i.e., $2 \le s < t \le m-2$, then (ii-a-1) when $s = 2 < t \le m-2$ or $2 \le s < t = m-2$, from Lemma 3.5(I)(ii)(c) or (d), we have $w_1 = 0$, where w_1 is given in Lemma 3.3, and hence there does not exist an array since all the factorial effects up to the two-factor interactions are estimable, and (ii-a-2) when $3 \le s < t \le m-3$, from Lemma 3.5(I)(ii)(e), we get Table 3.2(iii), where $w_1 \ne 0$, $s \ne m/2$ and $t \ne m/2$, and (ii-b) if rank$\{K^*_2\} = 1$, i.e., $s = 1$ and $2 \le t \le m-2$ or $2 \le s \le m-2$ and $t = m-1$, then from Lemma 3.5(I)(ii)(a) or (b), we get $w_1 \ne 0$ and $w_2 \ne 0$, where put $u = t$ in w_2 given in Lemma 3.3 when $s = 1$ and $2 \le t \le m-2$, and put $u = s$ when $2 \le s \le m-2$ and $t = m-1$, and hence Table 3.2(iv)(a) and (b). (B) If rank$\{K^*_0\} = 3$, i.e., $\lambda_i \ne 0$ $(i = p, q, r; 0 \le p < q < r \le m)$ and $\lambda_j = 0$ $(j \ne p, q, r; 0 \le j \le m)$, then from Lemma 3.3, $\tilde{w}_0 = w_0 = 0$ hold. (i) If $p = 0 < q < r \le m$ or $0 \le p < q < r = m$, then from Lemma 3.5(II)(ii)(a) or (b), we have $w^*_0 \ne 0$, $\tilde{w}_1 = 0$, $w_1 \ne 0$, rank$\{K^*_1\} = 2$ and rank$\{K^*_\gamma\} = 4 - \gamma$ $(2 \le \gamma \le 3)$, where put $(s,t) = (q,r)$ in $\tilde{w}_1$ and w_1 given in Lemma 3.3 when $p = 0 < q < r \le m$, and put $(s,t) = (p,q)$ when $0 \le p < q < r = m$, and hence we get Table 3.2(v)(a) and (b). (ii) If $p = 1 < q < r \le m-1$ or $1 \le p < q < r = m-1$, then from Lemma 3.5(II)(ii)(c) and (d), or (e) and (f), we have $w^*_0 \ne 0$ and rank$\{K^*_\gamma\} = 4 - \gamma$ $(1 \le \gamma \le 3)$, and hence Table 3.2(vi)(a) and (b). (iii) If $2 \le p < q < r \le m-2$, then from Lemma 3.5(II)(ii)(g), rank$\{K^*_\gamma\} = 4 - \gamma$ $(1 \le \gamma \le 3)$ hold, where $0 \le p < (m - \sqrt{m})/2$

and $w_0^* \neq 0$, and hence we get Table 3.2(vii). It follows from Lemmas 3.1 and 3.3 that the sufficient condition can be easily proved.

(III) Let T be of resolution R$(\{1,2\}|3)$, then rank$\{K_0^*\} = 3$ holds, and hence $\lambda_i \neq 0$ $(i = p, q, r; 0 \leq p < q < r \leq m)$ and $\lambda_j = 0$ $(j \neq p, q, r; 0 \leq j \leq m)$, and in addition $\tilde{w}_0 = w_0^* = 0$ and $w_0 \neq 0$ hold. From Lemma 3.5(II)(iii), there does not exist an array with $p = 0$ and $1 \leq q < r \leq m$ or $p = 1$ and $2 \leq q < r \leq m$. Thus from Proposition 2.4, we only consider $2 \leq p < q < r \leq m - 2$. In this cases, we have rank$\{K_\gamma^*\} = 4 - \gamma$ $(1 \leq \gamma \leq 3)$, and hence we get Table 3.3(i), where $w_0 \neq 0$. From Lemmas 3.1 and 3.3, the sufficient condition can be easily shown.

(IV) Let T be of resolution R$(\{0,1,2\}|3)$, then (A) if rank$\{K_0^*\} = 4$, and in addition (i) if rank$\{K_1^*\} = 3$, and furthermore (i-a) if rank$\{K_2^*\} = 2$, then from Proposition 2.3, we get $K_3^* = 0$. Thus we have (i-a-1) $\lambda_0 + \lambda_m \neq 0$, $\lambda_i \neq 0$ $(i = 1, 2, m-2)$ and $\lambda_j = 0$ $(j \neq 0, 1, 2, m-2, m; 3 \leq j \leq m-1)$, (i-a-2) $\lambda_0 + \lambda_m \neq 0$, $\lambda_i \neq 0$ $(i = 2, m-2, m-1)$ and $\lambda_j = 0$ $(j \neq 0, 2, m-2, m-1, m; 1 \leq j \leq m-3)$, and (i-a-3) $\lambda_0 \geq 0$, $\lambda_i \neq 0$ $(i = 1, 2, m-2, m-1)$, $\lambda_m \geq 0$ and $\lambda_j = 0$ $(j \neq 0, 1, 2, m-2, m-1, m; 3 \leq j \leq m-3)$, and hence we get Table 3.4(i)(a), (b) and (ii). (i-b) If rank$\{K_2^*\} = 1$, i.e., $\lambda_0 + \lambda_m \neq 0$, $\lambda_1 \neq 0$, $\lambda_u \neq 0$ $(2 \leq u \leq m-2)$, $\lambda_{m-1} \neq 0$ and $\lambda_j = 0$ $(j \neq 0, 1, u, m-1, m; 2 \leq j \leq m-2)$, then from Lemma 3.3, $w_2 = 0$ holds. Thus we have $u = n$, where $m = 2n$, and hence $K_3^* = 0$. Therefore we get Table 3.4(iii). (ii) If rank$\{K_1^*\} = 2$, i.e., $\lambda_i \neq 0$ $(i = 0, s, t, m; 1 \leq s < t \leq m-1)$ and $\lambda_j = 0$ $(j \neq 0, s, t, m; 1 \leq j \leq m-1)$, then from Lemma 3.3, $\tilde{w}_1 = w_1 = 0$ hold. Thus from Lemma 3.5(I)(iii), we get Table 3.4(iv). In particular, if $k = 2$, i.e., $m = 6$, then we have $s = 2$ and $t = 4$, and hence rank$\{K_2^*\} = 2$ and $K_3^* = 0$, and if $k \geq 3$, then rank$\{K_\gamma^*\} = 4 - \gamma$ $(2 \leq \gamma \leq 3)$. (B) If rank$\{K_0^*\} = 3$, i.e., $\lambda_i \neq 0$ $(i = p, q, r; 0 \leq p < q < r \leq m)$ and $\lambda_j = 0$ $(j \neq p, q, r; 0 \leq j \leq m)$, then from Lemma 3.3, we have $\tilde{w}_0 = w_0 = w_0^* = 0$. Thus from Lemma 3.5(II)(iv), we get Table 3.4(v). In particular, if $k = 4$, we have $p = 1$, $q = 3$, $r = 5$, $w_2 = 0$, rank$\{K_\gamma^*\} = 4 - \gamma$ $(\gamma = 1, 3)$ and rank$\{K_2^*\} = 1$, and if $k \geq 5$, then rank$\{K_\gamma^*\} = 4 - \gamma$ $(1 \leq \gamma \leq 3)$. Sufficient conditions can be easily obtained.

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References

Chakravarti, I. M. (1956). Fractional replication in asymmetrical factorial designs and partially balanced arrays, *Sankhyā*, **17**, 143–164.

Ghosh, S. and Kuwada, M. (2001). Estimable parametric functions for balanced fractional 2^m factorial designs, *Statistical Research Group*, TR **01–7**, Hiroshima University, Japan.

Hyodo, Y. (1989). Structure of fractional factorial designs derived from two-symbol balanced arrays and their resolution, *Hiroshima Math. J.*, **19**, 457–475.

Kuwada, M. and Yanai, T. (1998). On some estimable parametric functions for balanced fractional 2^m factorial designs, *Jr. Comb., Inf. & Syst. Sci.*, **23**, 17–32.

Kuwada, M., Hyodo, Y. and Yumiba, H. (2002). GA-optimal balanced fractional 2^m factorial designs of resolution $\mathrm{R}^*(\{0,1\}|3)$, submitted.

Shirakura, T. (1976). Optimal balanced fractional 2^m factorial designs of resolution VII, $6 \leq m \leq 8$, *Ann. Statist.*, **4**, 515–531.

Shirakura, T. (1980). Necessary and sufficient condition for a balanced array of strength 2ℓ to be a balanced fractional 2^m factorial design of resolution 2ℓ, *Austral. J. Statist.*, **22**, 69–74.

Shirakura, T. and Kuwada, M. (1975). Note on balanced fractional 2^m factorial designs of resolution $2\ell+1$, *Ann. Inst. Statist. Math.*, **27**, 377–386.

Shirakura, T. and Kuwada, M. (1976). Covariance matrices of the estimates for balanced fractional 2^m factorial designs of resolution $2\ell+1$, *J. Japan Statist. Soc.*, **6**, 27–31.

Srivastava, J. N. (1970). Optimal balanced 2^m fractional factorial designs, *Essays in Probability and Statistics*, (Eds. Bose, R. C. *et al.*), 689–706, The University of North Carolina Press, Chapel Hill.

Srivastava, J. N. and Chopra, D. V. (1971). On the characteristic roots of the information matrix of 2^m balanced factorial designs of resolution V, with applications, *Ann. Math. Statist.*, **42**, 722–734.

Yamamoto, S. and Hyodo, Y. (1984). Extended concept of resolution and the designs derived from balanced arrays, *TRU Math.*, **20**, 341–349.

Yamamoto, S., Shirakura, T. and Kuwada, M. (1975). Balanced arrays of strength 2ℓ and balanced fractional 2^m factorial designs, *Ann. Inst. Statist. Math.*, **27**, 143–157.

Yamamoto, S., Shirakura, T. and Kuwada, M. (1976). Characteristic polynomials of the information matrices of balanced fractional 2^m factorial designs of higher $(2\ell+1)$ resolution, *Essays in Probability and Statistics*, (Eds. Ikeda, S. *et al.*), 73–94, Shinko Tsusho, Tokyo.