A CHARACTERIZATION OF MODEL APPROACH FOR GENERATING BIVARIATE LIFE DISTRIBUTIONS USING REVERSED HAZARD RATES

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This paper aims at balancing between characterization and modeling approaches for bivariate extensions of univariate life distributions using reversed hazard rates. The proposed model, a characterized one, admits important properties irrespective of the choice of the univariate marginals. The retention of a univariate class property has been ensured along with a result on parallel combination of a bivariate system.

Key words and phrases: Characterized model, Distribution function, Life distribution class property, Parallel combination, Reversed hazard rates.

1. Introduction

The problem of generating bivariate life distributions from univariate ones is drawing the attention of the reliability analysts for quite long. As a result, several approaches have been developed in the literature for generalizing univariate laws. Amongst those approaches, the characterization approach and the modeling approach are very appealing. In fact, characterization approach is of interest to both theoreticians and applied workers. However, a few basic problems arise when one extends univariate distributions to multivariate forms through characterization approach. Firstly, there is a definitional problem in multivariate extensions of univariate measures. For example, multivariate failure rate has been defined in the literature in three different ways (see Basu (1971); Johnson and Kotz (1975); Shanbhag and Kotz (1987)). This means, one has to make a subjective choice if a property is to be used in terms of multivariate failure rate. As a result, uniqueness of the characterization approach becomes highly questionable. The other problem is the problem of generalization. It arises out of multiple ways that a univariate characterizing property can be extended in higher dimensions. Resultant distribution can hardly be unique. Another problem is the problem of selecting a characterization property because there are mostly a large number of important properties. Different properties may lead to different multivariate forms. In view of the above, it becomes hardly a reality to uniquely define the multivariate form of a univariate distribution. Further, the characterization results are distribution dependent and have limited appeals.

The other important approach is the modeling approach. It uses functional equations to determine a multivariate distribution from univariate marginal distributions. But there can be many such equations. Morgenstern (1956), Farlie
(1960) and Kelker (1970) proposed models from different directions. Kelker’s spherical and elliptical models suffer from the drawback that the underlying univariate distribution must be identical and symmetric in nature.

The major limitation of all the above mentioned models lies in their inability to retain the basic life distribution class properties. The Multivariate Extension Model of Roy and Mukherjee (1998) can preserve only the IFR and IFRA properties.

Thus a need for developing a general approach may be felt so that one can suitably eliminate the limitations of earlier approaches. To this end, we present a combined procedure, a characterized model. This is a distribution free characterization of a model to ensure retention of some univariate class properties. In the next section we introduce the proposed model. In section 3 we present some general bivariate properties. In that process we show that if both the marginal distributions belong to Decreasing Reversed Hazard Rate (DRHR) class then the modeled bivariate distribution belongs to Bivariate Decreasing Reversed Hazard Rate (BDRHR) class.

2. Characterization of the model

Let us denote by $X = (X_1, X_2)$ two nonnegative component lives, $X_1$ and $X_2$. Let $F_i(x_i)$ be the marginal distribution function of $X_i$, and $f_i(x_i)$ be the density function, $i = 1, 2$. By definition, the Reversed Hazard Rate (RHR), $a_i(x_i)$, and the cumulative RHR, $A_i(x_i)$, are

$$a_i(x_i) = \frac{f_i(x_i)}{F_i(x_i)}, \quad A_i(x_i) = \int_{x_i}^{\infty} a_i(u_i) du_i.$$  \hspace{1cm} (2.1)

Therefore,

$$F_i(x_i) = \exp\{-A_i(x_i)\}, \quad \frac{d}{dx_i} A_i(x_i) = -a_i(x_i), \quad i = 1, 2.$$  \hspace{1cm} (2.2)

Making use of these marginal distributions of $X_1$ and $X_2$ let us present a Characterized Extension Model (CE Model) based on the following thoughts.

**Basic thoughts**: At the time of extending a univariate life distribution to a higher dimension one must retain the basic features of the original life distributions. Since the concept of RHR plays an important role in reliability theory, as may be seen from the works of Keilson and Sumita (1982), Shaked and Shanthikumar (1994), Sengupta and Nanda (1997), Block *et al.* (1998), and Chandra and Roy (2001), we would like to incorporate in the bivariate model the retention of univariate RHR structure. Thus, functional forms of $a_1(x_1)$ and $a_2(x_2)$ are the two basic structures that one should retain in the bivariate system and this retention should logically be in terms of the corresponding bivariate RHRs. Let $F(x_1, x_2)$ be the distribution function of $X = (X_1, X_2)$. We propose that the corresponding bivariate RHRs be defined by

$$a_i(x_1, x_2) = -\frac{\delta}{\delta x_i} A(x_1, x_2), \quad i = 1, 2.$$  \hspace{1cm} (2.3)
where
\[(2.4) \quad A(x_1, x_2) = -\log F(x_1, x_2)\]
is the Bivariate Cumulative RHR function.

Thus, \(a_i(x_1, x_2)\) must retain the structure of \(a_i(x_i), i = 1, 2\). In other words, \(a_i(x_1, x_2)\) must be locally proportional to \(a_i(x_i)\) where the constant of proportionality, \(c_i(x_{3-i})\), may depend only on \(x_{3-i}\). The following theorem establishes that this above consideration uniquely determines a model, which will be referred as CE model for our subsequent discussion.

**Theorem 1.** Bivariate RHRs are locally proportional to the corresponding univariate RHRs if and only if the bivariate distribution function is of the form
\[(2.5) \quad F(x_1, x_2) = F_1(x_1)F_2(x_2)\exp\{-\gamma(\log F_1(x_1))(\log F_2(x_2))\}\]
for some \(\gamma\).

**Proof.** Under (2.5) it is easy to note that
\[a_1(x_1, x_2) = \{1 + \gamma A_2(x_2)\}a_1(x_1), a_2(x_1, x_2) = \{1 + \gamma A_1(x_1)\}a_2(x_2).\]
Thus, bivariate RHRs are locally proportional to the corresponding univariate RHRs.

To prove the converse let bivariate RHRs be locally proportional. With proportionality constants as \(c_1(x_2)\) and \(c_2(x_1)\) we write
\[(2.6) \quad a_1(x_1, x_2) = c_1(x_2)a_1(x_1), \quad a_2(x_1, x_2) = c_2(x_1)a_2(x_2).\]
Using the concept of line integration, two equivalent expressions of the underlying distribution function are obtained as
\[(2.7) \quad F(x_1, x_2) = \exp\left\{-\int_{x_1}^{\infty} a_1(u, \infty)du - \int_{x_2}^{\infty} a_2(x_1, v)dv\right\}\]
and
\[(2.8) \quad F(x_1, x_2) = \exp\left\{-\int_{x_1}^{\infty} a_1(u, x_2)du - \int_{x_2}^{\infty} a_2(\infty, v)dv\right\}\]
Comparing (2.7) and (2.8) we get for all \(x_1, x_2(\geq 0)\)
\[(2.9) \quad c_1(\infty)A_1(x_1) + c_2(\infty)A_2(x_2) = c_1(x_2)A_1(x_1) + c_2(x_1)A_2(x_2).\]
It may be observed from (2.9) that the left hand side is linear in \(A_2(x_2)\). Hence the right hand side of (2.9) must be linear in \(A_2(x_2)\). This implies that
\[(2.10) \quad c_1(x_2) = \alpha + \gamma A_2(x_2),\]
where \(\alpha = c_1(\infty) = 1\) because \(A_2(\infty) = 0\). Now, simplifying (2.8) by using (2.10) and using the fact that \(c_2(\infty) = 1\), we have
\[(2.11) \quad F(x_1, x_2) = \exp\{-A_1(x_1) - A_2(x_2) - \gamma A_1(x_1)A_2(x_2)\}.
This ensures (2.5).

It may be noted from the above result that the form of the characterized model given at (2.5) can be used for bivariate extension of univariate distributions retaining univariate structure of RHRs. Let us examine the set of values of $\gamma$ for which (2.5) is a proper distribution function. The following result provides an answer to this problem by specifying the range of values for $\gamma$.

**Theorem 2.** The distribution function $F(x_1, x_2)$ defined through the CE Model at (2.5) is defined if and only if $0 \leq \gamma \leq 1$. The corresponding joint density function is given by

$$f(x_1, x_2) = F_1(x_1)F_2(x_2)a_1(x_1)a_2(x_2)[\{1 + \gamma A_1(x_1)\}{1 + \gamma A_2(x_2)} - \gamma]$$

$$\cdot \exp[-\gamma(\log F_1(x_1))(\log F_2(x_2))].$$

**Proof.** To prove that $F(x_1, x_2)$ given by (2.5) is a proper distribution function we need to ensure as necessary and sufficient condition.

(i) $F(x_1, 0) = F(0, x_2) = 0$,

(ii) $F(\infty, \infty) = 1$, and

(iii) $(\delta^2/\delta x_1\delta x_2)F(x_1, x_2) \geq 0, \forall x_1, x_2.$

It is easy to verify (i) and (ii) from the properties of the marginal distribution functions $F_i(x_i), i = 1, 2$. For (iii), we observe that

$$\frac{\delta^2}{\delta x_1\delta x_2}F(x_1, x_2) = F_1(x_1)F_2(x_2)\exp(-\gamma A_1(x_1)A_2(x_2))$$

$$\cdot a_1(x_1)a_2(x_2)[\{1 + \gamma A_1(x_1)\}{1 + \gamma A_2(x_2)} - \gamma],$$

which gives the density function as in (2.12) provided the condition of nonnegativity holds. What we need to prove is that the condition $0 \leq \gamma \leq 1$ is both necessary and sufficient for (iii) to hold. To prove the necessary part let us assume, if possible, $\gamma < 0$. In that case in the expression (2.13) all the terms are positive except

$$(1 + \gamma A_1(x_1))(1 + \gamma A_2(x_2)) - \gamma.$$

Since $A_1(x_1)$ is continuous (as $a_1(x_1)$ exists) and varies decreasingly from $\infty$ to 0 we can choose an $x_1^*$ such that $A_1(x_1^*) > 1 - 1/\gamma(> 0)$. With such a choice of $x_1^*$, as $x_2 \to \infty$ the expression at (2.14) becomes negative. This happens because $\gamma$ is negative and $A_2(x_2) \to 0$ as $x_2 \to \infty$. Then (2.13) also becomes negative, which is a contradiction. Thus, $\gamma$ must be greater than 0. Next consider the case $\gamma > 1$. With a choice of $x_1 = x_2 = \infty$ we observe that the expression at (2.14) becomes negative as $A_1(\infty) = A_2(\infty) = 0$. As a result (2.13) becomes negative either at the point $(\infty, \infty)$ or in a neighbourhood of it. There is again a contradiction, which implies that $\gamma \leq 1$. Combining these two results we get that (iii) holds only if $0 \leq \gamma \leq 1$.

To prove the sufficiency part, we note that for $0 \leq \gamma \leq 1$,

$$(1 + \gamma A_1(x_1))(1 + \gamma A_2(x_2)) \geq 1.$$
as $A_1(x_1) \geq 0$, $A_2(x_2) > 0$. Hence the expression at (2.14) is greater than or equal to $(1 - \gamma)$ which is nonnegative. Since the rest of the terms in (2.13) are nonnegative, (iii) holds. Thus, $0 \leq \gamma \leq 1$ is both necessary and sufficient condition for the validity of the expression (2.5) as a bivariate distribution function.

Given the Theorems 1 and 2, it is easy to verify that (2.9) admits $F_i(x_i)$ as the distribution function of $X_i$, $i = 1, 2$. Thus, CE model is a suitable model for bivariate extension from univariate distributions. It may also be noted that $\gamma = 0$ implies independence of $X_1$ and $X_2$. Thus, an estimator of $\gamma$ can be used for testing independence under this model.

From the CE model one may obtain Morgenstern (1956)'s model as an approximation. According to CE model

$$F(x_1, x_2) = F_1(x_1)F_2(x_2)\exp[-\gamma\{\log F_1(x_1)\}{\log F_2(x_2)}].$$

Approximating

$$\exp[-\gamma\{\log F_1(x_1)\}{\log F_2(x_2)}]$$

in the following way:

$$\exp[-\gamma\{\log(1 - (1 - F_1(x_1)))\}{\log(1 - (1 - F_2(x_2)))}]$$

$$\simeq \exp[-\gamma\{1 - F_1(x_1)\}{1 - F_2(x_2)}]$$

$$\simeq [1 - \gamma\{1 - F_1(x_1)\}{1 - F_2(x_2)}]$$

we note, for $\theta = -\gamma$, a similarity between (2.5) and Morgenstern (1956)'s model

$$F(x_1, x_2) = F_1(x_1)F_2(x_2)[1 + \theta(1 - F_1(x_1))(1 - F_2(x_2))].$$

A method for generation of random observations from the CE model is given below. The conditional distribution function of $X_1$ given $X_2$ works out as

$$(2.15) \quad F(x_1 \mid x_2) = (1 + \gamma A_1(x_1))\exp\{-A_1(x_1)(1 + \gamma A_2(x_2))\}.$$
is \( F(x) = \exp(-\gamma/x) \) so that \( \{1/X\} \) follows an exponential distribution. For two marginal inverse exponential distributions, we note that cumulative RHRs are \( \lambda_1/x_1 \) and \( \lambda_2/x_2 \). Then, from Theorem 1 we get the Bivariate Inverse Exponential Distribution as the following:

\[
F(x_1, x_2) = \exp \left\{ -\frac{\lambda_1}{x_1} - \frac{\lambda_2}{x_2} - \gamma \frac{\lambda_1}{x_1} \frac{\lambda_2}{x_2} \right\}.
\]

Similarly, one may generalize the Inverse Rayleigh, Inverse Weibull distributions.

3. Some important properties

Our next theorem shows that the CE model retains Bivariate DRHR class properties when similar univariate class properties hold good for both the marginals.

**Theorem 3.** Under CE model \( X_1 \) and \( X_2 \) are marginally DRHR if and only if they are jointly BDRHR.

**Proof.** Under CE Model the \( A(x_1, x_2) \) is given by

\[
A(x_1, x_2) = A_1(x_1) + A_2(x_2) + \gamma A_1(x_1)A_2(x_2)
\]

and hence bivariate RHRs, \( a_i(x_1, x_2), i = 1, 2 \) are

\[
a_i(x_1, x_2) = \{1 + \gamma A_{3-i}(x_{3-i})\}a_i(x_i).
\]

From (3.2) we observe that \( a_i(x_i) \) is decreasing in \( x_i \) if and only if \( a_i(x_1, x_2) \) is decreasing in \( x_i \) and this is true for \( i = 1, 2 \). Combining these two necessary and sufficient conditions we note that \( X = (X_1, X_2) \) is BDRHR if and only if each of \( X_1 \) and \( X_2 \) is DRHR.

Not only the retention of class properties but also the retention of a few important results under independence can be ensured for the CE model

**Theorem 4.** Under the CE model if component lives have DRHR property then the parallel combination of those components has also DRHR property.

**Proof.** Writing \( Z \) as the life of the parallel combination of component lives \( X_1, X_2 \) jointly following CE model, we have the distribution function of \( Z \) as

\[
F_Z(z) = P[X_1 \leq z, X_2 \leq z] = \exp \{-A_1(z) - A_2(z) - \gamma A_1(z)A_2(z)\}.
\]

Thus, the RHR of \( Z \) is given by

\[
a_z(z) = a_1(z)(1 + \gamma A_2(z)) + a_2(z)(1 + \gamma A_1(z)).
\]

Since \( 0 \leq \gamma \leq 1, A_i(z) > 0 \forall z, i = 1, 2 \) and both \( a_i(z) \) and \( A_i(z) \) are decreasing in \( z, i = 1, 2 \), we have \( a_z(z) \) as a decreasing function in \( z \). Thus, \( Z = \text{Max}(X_1, X_2) \) has also DRHR property.
Another interesting observation regarding this model is the retention of the functional form under a bivariate parallel combination of independent prototypes. Writing $X_i = (X_{1i}, X_{2i})$ as the life vector of the $i$-th prototype, $i = 1, 2, \ldots, k$, let us define the bivariate parallel combination as a collection of two parallel combinations, the first one with all the first components and the second one with all the second components. Thus, the bivariate life vector of this parallel combination is given by $Z = (Z_1, Z_2)$ where

$$Z_1 = \text{Max}\{X_{1i}\}, \quad Z_2 = \text{Max}\{X_{2i}\}.$$ 

The corresponding distribution function of $Z$ simplifies to

$$F_Z(z_1, z_2) = [F_1(z_1)]^k[F_2(z_2)]^k \exp\{(-\gamma/k)(\log(F_1(z_1))^k)(\log(F_2(z_2))^k)\}.$$ 

This implies an adherence to CE model and in this sense we have a closure property of CE model under a bivariate parallel combination.

In view of the properties presented in the current section we may claim that the CE model has many advantages over other standard models available in the literature. The corresponding multivariate model with similar properties can be easily obtained.

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References