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ESTIMATIONS OF THE PARAMETERS OF THE WEIBULL DISTRIBUTION WITH PROGRESSIVELY CENSORED DATA

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We obtained estimation results concerning a progressively type-II censored sample from a two-parameter Weibull distribution. The maximum likelihood method is used to derive the point estimators of the parameters. An exact confidence interval and an exact joint confidence region for the parameters are constructed. A numerical example is presented to illustrate the methods proposed here.

Key words and phrases: Confidence interval, Joint confidence region, Maximum likelihood estimator, Progressively type-II censored sample.

1. Introduction

The Weibull distribution is widely used in reliability analysis. The probability density function of a two-parameter Weibull distribution has the form:

$$f(x) = \begin{cases} \dfrac{\nu}{\beta}\left(\dfrac{x}{\beta}\right)^{\nu-1} \exp\left\{-\left(\dfrac{x}{\beta}\right)^{\nu}\right\}, & \text{for} \quad x > 0, \\ 0, & \text{elsewhere,} \end{cases} \tag{1.1}$$

where $\beta > 0$ is a scale parameter and $\nu > 0$ is a shape parameter. For $0 < \beta < 1$, the Weibull distribution has a decreasing hazard function. With $\beta > 1$, the Weibull distribution has an increasing hazard function. When $\beta = 1$, the Weibull reduces to an exponential distribution. Note that the parameters β and ν are dependent through the equation:

$$\log\beta + \frac{1}{\nu}\log 2 = \log(\text{median}).$$

In this study, we consider a censoring scheme called progressive type-II censoring. Under this scheme, n units are placed on a test at time zero, with m failures to be observed. When the first failure is observed, r_1 of the surviving units are randomly selected and removed. At the second observed failure, r_2 of the surviving units are randomly selected and removed. This experiment stops at the time when the m-th failure is observed and the remaining $r_m = n - r_1 - r_2 - \cdots - r_{m-1} - m$ surviving units are all removed. The statistical inference for the Weibull distribution under progressive type-II censoring has been investigated by several authors such as Mann (1969, 1971), Cohen (1975),

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Cacciari and Montanari (1987), Wong (1993), Viveros and Balakrishnan (1994), Balasooriya *et al.* (2000), and Balakrishnan and Aggarwala (2000).

In this article, we consider progressively type-II censored data from a two-parameter Weibull distribution. We obtain the maximum likelihood estimators of the parameters in Section 2. In Section 3, we derive an exact confidence interval for the parameter ν and an exact joint confidence region for the parameters ν and β. A numerical example is presented for illustration in Section 4.

2. Point estimations of parameters

In this section, the maximum likelihood estimators (MLEs) for the parameters of the Weibull distribution based on progressive type-II censoring are derived. The probability density function of two-parameter Weibull distribution is given in (1.1), and corresponding cumulative distribution function is:

$$F(x) = 1 - \exp\left\{-\left(\frac{x}{\beta}\right)^{\nu}\right\}, \qquad x > 0.$$

Let $X_{1:m:n}, \ldots, X_{m:m:n}$ be a progressively type-II censored sample from a two-parameter Weibull distribution, with censoring scheme $\boldsymbol{r} = (r_1, \ldots, r_m)$. The likelihood function is given by

$$\begin{aligned} L(\nu, \beta) &= k \prod_{i=1}^{m} f(x_{i:m:n})[1 - F(x_{i:m:n})]^{r_i} \\ &= k\left(\frac{\nu}{\beta}\right)^{m} \prod_{i=1}^{m}\left(\frac{x_{i:m:n}}{\beta}\right)^{\nu-1} \exp\left\{-\sum_{i=1}^{m}(r_i+1)\left(\frac{x_{i:m:n}}{\beta}\right)^{\nu}\right\}, \end{aligned}$$

where

$$k = n(n-1-r_1)(n-2-r_1-r_2)\cdots(n-m+1-r_1-\cdots-r_{m-1}).$$

For simplicity of notation, we will use x_i instead of $x_{i:m:n}$. The log-likelihood function may then be written as

$$\log L(\nu, \beta) \propto m\log\nu - m\nu\log\beta + (\nu-1)\sum_{i=1}^{m}\log x_i - \sum_{i=1}^{m}(r_i+1)\left(\frac{x_i}{\beta}\right)^{\nu},$$

and hence we have the likelihood equations for β and ν to be

$$\text{(2.1)} \qquad \frac{\partial \log L}{\partial \beta} = -\frac{m\nu}{\beta} + \nu\beta^{-(\nu+1)}\sum_{i=1}^{m}(r_i+1)x_i^{\nu} = 0,$$

and

$$\text{(2.2)} \qquad \frac{\partial \log L}{\partial \nu} = \frac{m}{\nu} - m\log\beta + \sum_{i=1}^{m}\log x_i - \sum_{i=1}^{m}(r_i+1)\left(\frac{x_i}{\beta}\right)^{\nu}\log\left(\frac{x_i}{\beta}\right) = 0.$$

The MLEs $\hat{\beta}$ and $\hat{\nu}$ can be obtained by solving the likelihood equations. Equation (2.1) yields the MLE of β to be

$$\hat{\beta} = \left\{ \frac{1}{m} \sum_{i=1}^{m} (r_i + 1) x_i^{\hat{\nu}} \right\}^{1/\hat{\nu}} . \tag{2.3}$$

Equation (2.2), in conjunction with the MLE of β in (2.3), reduces to

$$\frac{1}{\hat{\nu}} + \frac{1}{m} \sum_{i=1}^{m} \log x_i - \frac{\sum_{i=1}^{m} (r_i + 1) x_i^{\hat{\nu}} \log x_i}{\sum_{i=1}^{m} (r_i + 1) x_i^{\hat{\nu}}} = 0. \tag{2.4}$$

Since (2.4) cannot be solved analytically for $\hat{\nu}$, some numerical methods such as Newton's method must be employed.

3. Interval estimations of parameters

In this section, an exact confidence interval for ν and an exact joint confidence region for ν and β are investigated. Let $X_1 < \cdots < X_m$ denote a progressively type-II censored sample from a two-parameter Weibull distribution, with censoring scheme $\boldsymbol{r} = (r_1, \ldots, r_m)$. Further, let $Y_i = (X_i/\beta)^{\nu}$, $i = 1, \ldots, m$. It can be seen that $Y_1 < \cdots < Y_m$ is a progressively type-II censored sample from an exponential distribution with mean 1. Let us consider the following transformation:

$$\begin{cases} S_1 = nY_1, \\ S_2 = (n - r_1 - 1)(Y_2 - Y_1), \\ \quad \vdots \\ S_m = (n - r_1 - \cdots - r_{m-1} - m + 1)(Y_m - Y_{m-1}). \end{cases} \tag{3.1}$$

Thomas and Wilson (1972) proved that the generalized spacings $S_1, \ldots, S_m$, as defined in (3.1), are independent and identically distributed as an exponential distribution with mean 1. Hence,

$$V = 2S_1 = 2nY_1$$

has a chi-square distribution with 2 degrees of freedom and

$$U = 2\sum_{i=2}^{m} S_i = 2\left\{ \sum_{i=1}^{m} (r_i + 1) Y_i - nY_1 \right\}$$

has a chi-square distribution with $2m - 2$ degrees of freedom. We can also find that U and V are independent random variables. Let

$$T_1 = \frac{U}{(m-1)V} = \frac{\sum_{i=1}^{m} (r_i + 1) Y_i - nY_1}{n(m-1)Y_1}, \tag{3.2}$$

and

$$T_2 = U + V = 2\sum_{i=1}^{m}(r_i+1)Y_i. \tag{3.3}$$

It is easy to show that T_1 has an F distribution with $2m-2$ and 2 degrees of freedom and T_2 has a chi-square distribution with $2m$ degrees of freedom. Furthermore, by Johnson *et al.* (1994 p. 350), T_1 and T_2 are independent.

To obtain the confidence interval for ν and the joint confidence region for ν and β, we use the following lemma.

LEMMA 1. *Suppose that* $0 < a_1 < \cdots < a_m$. *Let*

$$T_1(\nu) = \frac{\sum_{i=1}^{m}(r_i+1)a_i^{\nu} - na_1^{\nu}}{n(m-1)a_1^{\nu}}.$$

Then, $T_1(\nu)$ *is strictly increasing in* ν *for any* $\nu > 0$. *Furthermore*, *if* $t > 0$, *the equation* $T_1(\nu) = t$ *has a unique solution for any* $\nu > 0$.

PROOF. The function $T_1(\nu)$ can be written as

$$T_1(\nu) = \frac{1}{n(m-1)}\sum_{i=1}^{m}(r_i+1)\left(\frac{a_i}{a_1}\right)^{\nu} - \frac{1}{m-1}.$$

Because $a_i/a_1 > 1$, $i = 2,\dots,m$, we have $(a_i/a_1)^{\nu}$ is an increasing function of ν, and hence $T_1(\nu)$ is strictly increasing in ν. Moreover, $\lim_{\nu\to\infty} T_1(\nu) = \infty$ and $\lim_{\nu\to 0} T_1(\nu) = 0$. Thus, if $t > 0$, $T_1(\nu) = t$ has a unique solution for any $\nu > 0$. □

Let $F_{\alpha(\delta_1,\delta_2)}$ be the percentile of F distribution with right-tail probability α and δ_1 and δ_2 degrees of freedom. The following theorem gives an exact confidence interval for the parameter ν.

THEOREM 1. *Suppose that* X_i, $i = 1,\dots,m$, *are the order statistics of a progressively type-II censored sample from a sample of size* n *from a two-parameter Weibull distribution*, *with censoring scheme* $(r_1,\dots,r_m)$. *Then a* $100(1-\alpha)\%$ *confidence interval for* ν *is*:

$$(\varphi(X_1,\dots,X_m,F_{1-(\alpha/2)(2m-2,2)}),\varphi(X_1,\dots,X_m,F_{(\alpha/2)(2m-2,2)})),$$

where $0 < \alpha < 1$ *and* $\varphi(X_1,\dots,X_m,t)$ *is the solution of* ν *for the equation*

$$\frac{\sum_{i=1}^{m}(r_i+1)X_i^{\nu} - nX_1^{\nu}}{n(m-1)X_1^{\nu}} = t.$$

PROOF. From (3.2), we know that the pivot

$$\begin{aligned} T_1 &= \frac{\sum_{i=1}^{m}(r_i+1)Y_i - nY_1}{n(m-1)Y_1} \\ &= \frac{\sum_{i=1}^{m}(r_i+1)X_i^{\nu} - nX_1^{\nu}}{n(m-1)X_1^{\nu}} \end{aligned}$$

has an F distribution with $2m-2$ and 2 degrees of freedom. Hence, for $0 < \alpha < 1$, the event

$$F_{1-(\alpha/2)(2m-2,2)} < \frac{\sum_{i=1}^{m}(r_i+1)X_i^{\nu} - nX_1^{\nu}}{n(m-1)X_1^{\nu}} < F_{(\alpha/2)(2m-2,2)}$$

is equivalent to the event

$$\varphi(X_1, \ldots, X_m, F_{1-(\alpha/2)(2m-2,2)}) < \nu < \varphi(X_1, \ldots, X_m, F_{(\alpha/2)(2m-2,2)}).$$

This completes the proof. □

Let us now discuss the joint confidence region for the parameters ν and β. A naive $100(1-\alpha)\%$ joint confidence region may be constructed by using B_α for (T_1, T_2) such that

$$B_\alpha = \{(t_1, t_2) \mid g_1(t_1)g_2(t_2) > c_\alpha\}, \qquad 0 < \alpha < 1,$$

(see Figure 1), where g_1 is the density function of F distribution, g_2 is the density function of chi-square distribution, and c_α is a constant given by

$$\begin{aligned} 1-\alpha &= \iint_{B_\alpha} g_1(t_1)g_2(t_2)\ dt_2\ dt_1 \\ &= \int_{h_1(\alpha)}^{h_2(\alpha)} g_1(t_1) \int_{\lambda(t_1|\alpha)}^{\xi(t_1|\alpha)} g_2(t_2)\ dt_2\ dt_1. \end{aligned}$$

Since T_1 and T_2 are independent, a simpler way is to choose ξ and λ depending only upon α such that $\int_{h_1(\alpha)}^{h_2(\alpha)} g_1(t_1)\ dt_1 = \sqrt{1-\alpha}$ and $\int_{\lambda(\alpha)}^{\xi(\alpha)} g_2(t_2)\ dt_2 = \sqrt{1-\alpha}$.

Let $\chi^2_{\alpha(\delta)}$ denote the percentile of chi-square distribution with right-tail probability α and δ degrees of freedom. An exact joint confidence region for the parameters ν and β is given in the following theorem.

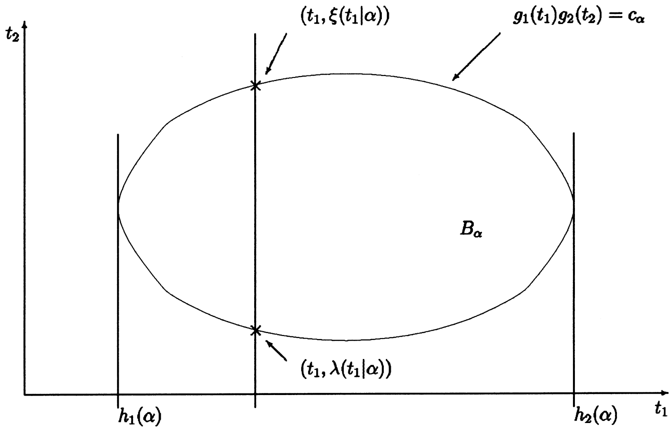

Figure 1. A naive confidence region.

THEOREM 2. *Suppose that X_i, $i = 1, \ldots, m$, are the order statistics of a progressively type-II censored sample from a sample of size n from a two-parameter Weibull distribution, with censoring scheme $(r_1, \ldots, r_m)$. Then a $100(1-\alpha)\%$ joint confidence region for ν and β is determined by the following inequalities:*

$$\begin{cases} \varphi(X_1, \ldots, X_m, F_{((1+\sqrt{1-\alpha})/2)(2m-2,2)}) \\ \qquad < \nu < \varphi(X_1, \ldots, X_m, F_{((1-\sqrt{1-\alpha})/2)(2m-2,2)}) \\ \left\{ \dfrac{2\sum_{i=1}^m (r_i+1) X_i^\nu}{\chi^2_{((1-\sqrt{1-\alpha})/2)(2m)}} \right\}^{1/\nu} < \beta < \left\{ \dfrac{2\sum_{i=1}^m (r_i+1) X_i^\nu}{\chi^2_{((1+\sqrt{1-\alpha})/2)(2m)}} \right\}^{1/\nu}, \end{cases}$$

where $0 < \alpha < 1$ and $\varphi(X_1, \ldots, X_m, t)$ is the solution of ν for the equation

$$\frac{\sum_{i=1}^m (r_i+1) X_i^\nu - nX_1^\nu}{n(m-1)X_1^\nu} = t.$$

PROOF. From (3.3), we know that

$$\begin{aligned} T_2 &= 2\sum_{i=1}^m (r_i+1) Y_i \\ &= 2\sum_{i=1}^m (r_i+1) \left(\frac{X_i}{\beta} \right)^\nu \end{aligned}$$

has a chi-square distribution with $2m$ degrees of freedom, and it is independent of T_1. Next, for $0 < \alpha < 1$, we have

$$P(F_{((1+\sqrt{1-\alpha})/2)(2m-2,2)} < T_1 < F_{((1-\sqrt{1-\alpha})/2)(2m-2,2)}) = \sqrt{1-\alpha},$$

and

$$P(\chi^2_{((1+\sqrt{1-\alpha})/2)(2m)} < T_2 < \chi^2_{((1-\sqrt{1-\alpha})/2)(2m)}) = \sqrt{1-\alpha}.$$

From these relationships, we obtain

$$\begin{aligned} P\Bigg(& F_{((1+\sqrt{1-\alpha})/2)(2m-2,2)} < \frac{\sum_{i=1}^m (r_i+1) X_i^\nu - nX_1^\nu}{n(m-1)X_1^\nu} < F_{((1-\sqrt{1-\alpha})/2)(2m-2,2)}, \\ & \chi^2_{((1+\sqrt{1-\alpha})/2)(2m)} < 2\sum_{i=1}^m (r_i+1) \left(\frac{X_i}{\beta} \right)^\nu < \chi^2_{((1-\sqrt{1-\alpha})/2)(2m)} \Bigg) \\ & = 1-\alpha. \end{aligned}$$

Equivalently,

$$\begin{aligned} P\Bigg(& \varphi(X_1, \ldots, X_m, F_{((1+\sqrt{1-\alpha})/2)(2m-2,2)}) \\ & < \nu < \varphi(X_1, \ldots, X_m, F_{((1-\sqrt{1-\alpha})/2)(2m-2,2)}), \\ & \left\{ \frac{2\sum_{i=1}^m (r_i+1) X_i^\nu}{\chi^2_{((1-\sqrt{1-\alpha})/2)(2m)}} \right\}^{1/\nu} < \beta < \left\{ \frac{2\sum_{i=1}^m (r_i+1) X_i^\nu}{\chi^2_{((1+\sqrt{1-\alpha})/2)(2m)}} \right\}^{1/\nu} \Bigg) = 1-\alpha. \end{aligned}$$

The theorem follows. □

4. Illustrative examples

To illustrate the use of the estimation methods proposed in this article, the following example is discussed.

Example 1. Nelson (1982, p. 228, table 6.1) presented data on the time to breakdown of an insulating fluid in an accelerated test conducted at various test voltages. In analyzing this data set, Nelson (1982) considered a Weibull distribution. For the purposes of illustrating the methods discussed in this article, a progressively type-II censored sample of size $m = 8$ was randomly selected from the $n = 19$ observations recorded at 34 kilovolts in Nelson's table, as given by Viveros and Balakrishnan (1994). The observations and censoring scheme are reported in Table 1.

Using the formulae described in Section 2, we obtain the MLEs of ν and β to be $\hat{\nu} = 0.9743$ and $\hat{\beta} = 9.2254$, respectively. To find a 95% confidence interval for ν, we need the percentiles

$$F_{0.025(14,2)} = 39.4265 \quad \text{and} \quad F_{0.975(14,2)} = 0.2059.$$

By Theorem 1 and using the Fortran IMSL nonlinear equation solver, the 95% confidence interval for ν is $(0.3242, 1.7692)$.

Table 1. Progressively type-II censored sample generated from the times to breakdown data.

i	1	2	3	4	5	6	7	8
x_i	0.19	0.78	0.96	1.31	2.78	4.85	6.50	7.35
r_i	0	0	3	0	3	0	0	5

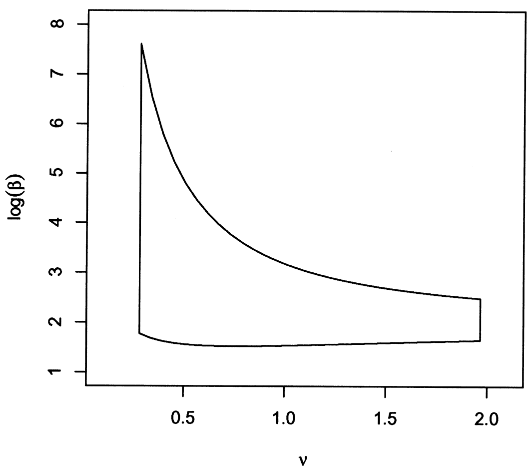

Figure 2. Joint confidence region for ν and $\log(\beta)$.

Furthermore, to obtain a 95% joint confidence region for ν and β, we need the following percentiles:

$$F_{0.0127(14,2)} = 78.4147, \quad F_{0.9873(14,2)} = 0.1648,$$
$$\chi^2_{0.0127(16)} = 31.2069 \quad \text{and} \quad \chi^2_{0.9873(16)} = 6.0684.$$

By Theorem 2, the 95% joint confidence region for ν and β is determined by the following inequalities:

$$\begin{cases} 0.2807 < \nu < 1.9648 \\ \left\{ \dfrac{2\sum_{i=1}^{8}(r_i+1)x_i^{\nu}}{31.2069} \right\}^{1/\nu} < \beta < \left\{ \dfrac{2\sum_{i=1}^{8}(r_i+1)x_i^{\nu}}{6.0684} \right\}^{1/\nu} \end{cases}.$$

In order to see the shape of this confidence region clearly, logarithmic scaling is recommended for scale parameter β. Figure 2 shows the 95% joint confidence region for ν and $\log(\beta)$. It is easy to see that the region is small when ν is large. □

5. Conclusions

We use the maximum likelihood method to obtain the point estimators of the parameters of a two-parameter Weibull distribution based on progressive Type II censoring. We provide two pivotal quantities to construct an exact confidence interval and an exact joint confidence region for the parameters, respectively. A numerical data set is analyzed in Section 4 to illustrate our approach.

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