ESTIMATION OF ASYMMETRICAL VOLATILITY FOR ASSET PRICES: THE SIMULTANEOUS SWITCHING ARIMA APPROACH

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Asymmetrical movements between the downward and upward phases of sample paths of many financial time series have been noted by economists. By incorporating the conditional heteroskedasticity aspect into the nonstationary simultaneous switching autoregressive (SSAR) model, the asymmetrical volatility function of financial time series with daily effects can be easily estimated. We report a simple empirical result for stock price daily indices of the Nikkei-225 and SP-500 using this model.

Key words and phrases: Asymmetrical Volatility, Stock Prices, Simultaneous Switching AR Model, Conditional Heteroskedasticity, Daily Effect.

1. Introduction

In the past few decades, several non-linear time series models have been proposed by econometricians. In particular, considerable attention has been paid to the autoregressive conditional heteroskedasticity (ARCH) model, which was originally proposed by Engle (1982). A number of extensions of the standard ARCH model have been proposed and some of them have been used in empirical studies. In addition, some non-parametric and semi-parametric estimation methods for conditional heteroskedasticities in asset prices and returns have been proposed. The related issues of modelling the conditional heteroskedasticities in asset prices and their volatilities have also been discussed by Bollerslev (1986) and Chapter 21 of Hamilton (1994), for instance.

In this paper we shall propose to use a new class of non-linear time series models called simultaneous switching autoregressive (SSAR) models for analyzing stock prices and possibly other asset prices. In particular, we shall develop a non-stationary SSAR model with autoregressive conditional heteroskedastic (ARCH) disturbances in order to estimate the volatility functions of stock prices. The main reason for using this class of non-linear time series models is because it seems clear to that the class of autoregressive integrated moving-average (ARIMA) time series model and the standard ARCH models cannot describe one important aspect in some financial time series, that is, the asymmetrical movements in the upward phase (or regime) and in the downward phase (or regime). It has often been observed that major financial series including stock prices display some kind of asymmetrical movements in the upward and downward phases.
We estimate the volatility functions of asset prices, which can be asymmetrical in two phases, by the use of a non-stationary SSAR model with ARCH disturbances. Unlike other parametric, non-parametric, and semi-parametric methods already available, our formulation is a simple parametric approach which provides an easy way to handle the daily effects in the asymmetrical volatility functions of asset prices. Since these effects have been observed by some financial economists and practitioners in the financial markets, our method for estimating the volatility functions may have real applications. Furthermore, our estimation method for the simultaneous switching autoregressive integrated (SSARI) models can be justified by its asymptotic properties and hence the model selection procedure within a class of SSARI models based on the information criteria can be developed in a rather straightforward fashion.

Earlier, we introduced a simple stationary SSAR time series model and discussed its statistical properties in some detail (Kunitomo and Sato (1996, 2000), Sato and Kunitomo (1996)). In a subsequent work (Kunitomo and Sato (1999)), we extended the basic SSAR model, and introduced a class of simple non-stationary SSAR models. It is important to note that it is not possible to describe this kind of asymmetrical patterns in the upward and downward phases by using the standard linear non-stationary time series models, including the ARIMA time series model and the standard ARCH model proposed by Engle (1982). This issue has been previously pointed out by both Nelson (1991) and Harvey and Shephard (1996) in the context of the estimation problem of the volatility function of stock prices. Although there have been some proposed estimation methods for volatility functions, time series models often become complicated once the asymmetrical forms of the volatility functions are introduced. Since the stationary SSAR (simultaneous switching autoregressive) and non-stationary SSAR (simultaneous switching autoregressive) models are natural extensions of the corresponding AR and nonstationary AR models, they can easily be extended to handle asymmetrical conditional heteroskedasticities.

In Section 2, we shall introduce the univariate SSARI model with ARCH disturbances. We also investigate some properties of the SSARI($p$) model with a time trend and ARCH($r$) disturbances in some detail. We also discuss the asymptotic properties of the estimation method and develop the model selection procedure. Then in Section 3, we apply the SSARI model with a time trend for the analysis of the Nikkei 225 spot index in Japan and the SP 500 spot index in the United States. In Section 4, some concluding remarks on our econometric approach and empirical findings will be given. The proof of some theoretical results obtained in this paper will be in the Appendix.

2. The nonstationary SSAR model

There has been growing interest in the last few decades among econometricians to investigate financial time series data by using statistical time series analysis. Here we should briefly mention that there are several interesting features often observed in financial time series data.
First, many financial time series such as stock prices, bond prices, interest rates, and foreign exchange rates are often too volatile to use stationary time series models in standard statistical time series analysis. Second, the distributions of financial prices, yields, and returns are often not well approximated by the Gaussian distribution. It has often been found that the kurtosis calculated from the daily returns for stock prices is much larger than 3, for instance. However, there is little consensus on the class of distributions appropriate for describing financial time series among econometricians. Third, the estimated historical volatility functions for many financial time series are often not constant over time. Fourth, some financial time series sometimes exhibit asymmetrical movements in the upward phase as well as the downward phase. In particular, a number of economists have observed this type of asymmetrical time series movements in stock prices.

The standard linear time series models such as the autoregressive integrated moving average (ARIMA) process can help explain the first and second features, but not to the third and fourth features. The standard autoregressive conditional heteroskedasticity (ARCH) process, which was originally proposed by Engle (1982) and has been sometimes used in recent econometric applications, is consistent with the second and third features, but not with the fourth one. There have thus been several attempts to extend the standard ARCH model. (See Chapter 21 of Hamilton (1994), for instance.) Then time series modelling tends to become quite complicated in past studies.

2.1. An Example of Stock Price Model

We first introduce a simple econometric model for stock prices which lead us to develop the class of general non-stationary SSAR models. Let the intrinsic value of a security at time $t$ and its observed price be $V_t$ and $P_t$, respectively. We distinguish the intrinsic value of a security and its observed price in our formulation. Since two values $V_t$ and $P_t$ can be different, we can introduce a partial-adjustment model when the intrinsic value $V_t$ at $t$ deviates from its anticipated price $V_t^*$ in the market, which is given by

$$P_t - P_{t-1} = \begin{cases} g_1(V_t - V_t^*) & \text{if } V_t - V_t^* \geq 0 \\ g_2(V_t - V_t^*) & \text{if } V_t - V_t^* < 0 \end{cases} \tag{2.1}$$

We assume that the anticipated price $V_t^*$ of $V_t$ at $t$ using the past information is a linear combination of the past observed prices, which is given by

$$V_t^* = \sum_{i=0}^{p-1} \beta_i P_{t-i-1} \tag{2.2}$$

and the adjustment coefficients in the stock price equation satisfy the condition $g_i \geq 0$ ($i = 1, 2$). This is a modified version of the micro-market financial model proposed by Amihud and Mendelson (1987). We notice that if we use $P_{t+1} - P_t$ instead of $P_t - P_{t-1}$ in (2.1), the resulting model will be slightly different from (2.4).
The anticipated price $V_t^*$ in (2.2) includes the optimal forecasts of price levels given the past information if $P_t$ could have followed some ARIMA models. Because there are new shocks or news available at $t$ in markets, $V_t$ could be different from $V_t^*$. In addition, we have allowed the adjustment coefficients $g_i$ ($i = 1, 2$) to take different values. There could be intuitive economic reasons why they are different. For instance, when $V_t \geq V_t^*$ the current price has been under-evaluated and there is economic pressure mainly from the demand side to force the price up. On the other hand, when $V_t < V_t^*$ the current price has been over-evaluated and there is economic pressure mainly from the supply side to push the price down. Since there are two main forces during the actual price determination process in financial markets, the two coefficients $g_i$ ($i = 1, 2$) could be different.

When $g_1 = g_2$, (2.1) is reduced to the standard linear adjustment model. Further, when $g_1 = g_2 = 1$ and $V_t^* = P_{t-1}$, then $V_t = P_t$ and the intrinsic value of a security is always equal to its observed price. Hence, by using the formulation we have adopted in (2.1) it is possible to examine from the observed time series data if these conditions are reasonable descriptions of reality.

In recent financial economics, it has been often assumed that the intrinsic security value \{\$\} follows an integrated process $I(1)$ with a drift term,

\[
V_t = V_{t-1} + \sigma_t e_t + \mu,
\]

where $\mu$ and $\sigma_t$ represent the (constant) expected daily return and the volatility of the intrinsic value of the underlying security, respectively, and \{\$\} are a sequence of i.i.d. random variables generated by a stochastic process. We note that in the actual estimation we often transform the original data sets into their logarithms which has been a common practice in financial data analysis. Hence the underlying variables should be interpreted with some care in that case.

Let the indicator functions be $I_t^{(1)} = I(P_t \geq P_{t-1})$ and $I_t^{(2)} = I(P_t < P_{t-1})$, where $I(\omega) = 1$ if the event $\omega$ occurs and $I(\omega) = 0$ otherwise. By combining (2.3) with (2.1) and (2.2), we have the representation for $\Delta P_t$ as

\[
\Delta P_t = g(t) \left[ \frac{1}{g(t-1)} - \beta_0 \right] \Delta P_{t-1} - g(t) \sum_{i=1}^{p-1} \beta_i \Delta P_{t-1-i} + g(t)[\mu + \sigma_t e_t],
\]

where $g(t) = g_1 I_t^{(1)} + g_2 I_t^{(2)}$ and $\Delta$ is the difference operator such as $\Delta P_t = P_t - P_{t-1}$. In this representation, $I_t^{(1)} = 1$ if and only if $V_t - V_t^* \geq 0$. But then (2.1) implies that $I_t^{(1)} = 1$ if and only if $\Delta P_t \geq 0$. When $p = 1$, $\beta_0 = 1$ and $\sigma_t = \sigma$, then (2.4) is identical to the SSARI model in Kunitomo and Sato (1999).

2.2. The SSARI($p$)-ARCH($r$) model

We shall introduce a general class of the nonstationary simultaneous switching autoregressive (SSAR) model with autoregressive conditional heteroskedasticity (ARCH), which includes (2.4) as a special case. For the specific application
to the financial time series in this paper, we first consider the univariate SSAR model represented by

$$y_t = \begin{cases} 
  a_1^* + a_2^* t + \sum_{i=1}^{p} a_i y_{t-i} + \sigma_1 u_t \ & \text{(if } y_t \geq y_{t-1}) \\
  b_1^* + b_2^* t + \sum_{i=1}^{p} b_i y_{t-i} + \sigma_2 u_t \ & \text{(if } y_t < y_{t-1})
\end{cases},$$

(2.5)

where we take $\sigma_i > 0 \ (i = 1, 2)$.

Let $\mathcal{F}_{t-1}$ be the $\sigma$-field generated by a set of random variables $\{y_s, v_s; s \leq t-1\}$ and we consider the filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t \in \mathbb{Z}})$ ($\mathbb{Z}$ is the set of integers) as in the standard discrete time series analysis. We assume that the disturbance terms $\{u_t\}$ are a sequence of $I(1)$ process satisfying

$$\Delta u_t = v_t,$$

(2.6)

where the random variables $\{v_t\}$ are martingale differences with $E(v_t \mid \mathcal{F}_{t-1}) = 0$. The volatility function $h_t (= E(v_t^2 \mid \mathcal{F}_{t-1}))$ for the intrinsic value is given by

$$h_t = 1 + \sum_{i=1}^{r} \alpha_i v_{t-i}^2 \quad \text{a.s.,}$$

(2.7)

where the unknown coefficients $\{\alpha_i, i = 1, \ldots, r\}$ satisfy the restrictions on the positivity of conditional variances as $\alpha_i \geq 0 \ (i = 1, \ldots, r)$ and

$$0 \leq \sum_{i=1}^{r} \alpha_i < 1.$$

(2.8)

To sum up our formulation, the disturbance terms can be written as $v_t = w_t \sqrt{h_t}$, where we usually assume that $\{w_t\}$ is a sequence of i.i.d. random variables with $E(w_t) = 0$, $E(w_t^2) = 1$, and their density function $f(w)$ is positive everywhere with respect to the Lebesgue measure. We note that although the conditional heteroskedasticity function $h_t$ is symmetric, the resulting conditional heteroskedasticities for $\{y_t\}$ and $\{\Delta y_t\}$ can be asymmetrical. We can also introduce more complicated volatility models such as the class of the EGARCH models and we shall discuss this briefly in Section 3.

The univariate non-linear time series model we introduced in (2.5) has the first order multivariate autoregressive form by using the standard state space representation in time series analysis. Let us define $p \times 1$ vectors $\mathbf{y}_t$ and $\mathbf{u}_t$ by

$$\mathbf{y}_t = \begin{pmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-p+1} \end{pmatrix}, \quad \mathbf{u}_t = \begin{pmatrix} u_t \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

(2.9)

$p \times 2$ matrices $\mathbf{\mu}_i \ (i = 1, 2)$ by
\[ \mathbf{\mu}_1 = \begin{pmatrix} a_1^* & a_2^* \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}, \quad \mathbf{\mu}_2 = \begin{pmatrix} b_1^* & b_2^* \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}, \]

and \( p \times p \) matrices \( A \) and \( B \) by

\[ A = \begin{pmatrix} a_1 \cdots & \cdots & a_p \\ 1 & 0 & \cdots \\ \vdots & \vdots & \ddots \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \cdots & \cdots & b_p \\ 1 & 0 & \cdots \\ \vdots & \vdots & \ddots \\ 0 & 1 & 0 \end{pmatrix}. \]

Then the non-linear time series model we have introduced can be simply represented as

\[ y_t = \begin{cases} \mathbf{\mu}_1 z_t^* + Ay_{t-1} + \sigma_1 u_t & \text{if } e_t' y_t \geq e_t' y_{t-1} \\ \mathbf{\mu}_2 z_t^* + By_{t-1} + \sigma_2 u_t & \text{if } e_t' y_t < e_t' y_{t-1} \end{cases}, \]

where \( e_1' = (1, 0, \ldots, 0) \) and \( z_t^* = (1, t)' \) is the vector of strictly exogenous variables.

Although (2.12) is not in the Markovian form mathematically, it has the Markovian representation as (A.3) in the Appendix under the coherency conditions given by the following conditions. The most important feature of this representation is that the time series variables may take quite different values in two different phases or regimes. This type of statistical time series model could be classified as the threshold model in the recent time series literature. However, since the vector time series and two phases at time \( t \) are determined simultaneously, we shall refer this type of time series models to a class of simultaneous switching autoregressive (SSAR) time series model. The univariate time series model consisting of (2.5) and (2.6) can be called a simultaneous switching integrated autoregressive model. We denote this class of models as SSARI(\( p \)), which is a direct extension of the integrated AR(\( p \)) model in the statistical time series analysis. In the SSARI model, we need some restrictions on unknown parameters in order to make the stochastic process defined by (2.12) meaningful in a proper statistical sense. This issue has been called the coherency problem and extensively discussed in Kunitomo and Sato (1996, 1999, 2000). The terminology “coherency” has been adopted from Gourieroux et al. (1980) in econometrics. We say the non-linear time series model (2.12) is coherent if and only if the correspondence between \( \{y_t\} \) and \( \{u_t\} \) is one-to-one given the initial condition \( \mathcal{F}_0 \).

The conditions that \( e_t' y_t \geq e_t' y_{t-1} \) and \( e_t' y_t < e_t' y_{t-1} \) can be rewritten as

\[ \sigma_1 u_t \geq e_t'(I_p - A)y_{t-1} - e_t' \mathbf{\mu}_1 \]
and

\[ \sigma_2 u_t < e_1'(I_p - B) y_{t-1} - e_1' \mu_2, \]

respectively. Then the set of conditions on coherency in the present case can be summarized by a \(1 \times 2\) vector \(\gamma^* = (\gamma_1^*, \gamma_2^*)\) and a \(1 \times p\) vector \(\gamma' = (\gamma_1, \ldots, \gamma_p)\):

\[
\frac{1}{\sigma_1} [-e_1' \mu_1, e_1'(I_p - A)] = \frac{1}{\sigma_2} [-e_1' \mu_2, e_1'(I_p - B)] = (\gamma^*, \gamma').
\]

Because the univariate SSARI\((p)\) model has some specific structure in the class of the general multivariate SSAR model, it has a simple representation. For instance, we do not need any additional conditions to (2.15) for the normalization and we have the one-to-one correspondence between the stochastic processes \(\{y_t\}\) and \(\{v_t\}\) under the condition given by (2.15).

In order to obtain a useful representation of the process \(\{\Delta y_t\}\), we use the indicator functions

\[
I_t^{(1)} = I(e_1' y_t \geq e_1' y_{t-1})
\]

and

\[
I_t^{(2)} = I(e_1' y_t < e_1' y_{t-1}).
\]

Then we can obtain the representation

\[
y_t = \mu(t) z_t^* + A(t) y_{t-1} + \sigma(t) u_t, \tag{2.16}
\]

where

\[
\mu(t) = \sum_{i=1}^{2} \mu_i I_t^{(i)},
\]

\[
A(t) = A I_t^{(1)} + B I_t^{(2)},
\]

and

\[
\sigma(t) = \sum_{i=1}^{2} \sigma_i I_t^{(i)}.
\]

When the first component of \(\{u_t\}\) in (2.16) is an \(I(1)\) process, the stochastic process \(\{y_t\}\) is a non-ergodic process. Hence it is of interest to investigate the conditions for the stationarity and ergodicity of the process \(\{\Delta y_t\}\). In the present univariate case we can simplify some coefficients by using the coherency conditions (2.15), that is, we have the relations

\[
\mu(t) = -(\gamma_1^*, \gamma_2^*) \sigma(t) \tag{2.17}
\]

and

\[
e_1' - e_1' A(t) = (\gamma_1, \ldots, \gamma_p) \sigma(t). \tag{2.18}
\]
From these relations, we re-write the disturbance terms \( \{u_t\} \) as

\[
(2.19) \quad u_t = \frac{1}{\sigma(t)} \Delta y_t + \gamma_1^* + \gamma_2^* t + \sum_{i=1}^{p} \gamma_i y_{t-i}.
\]

Because of (2.6) and (2.19), given the information available at \( t-1 \), we have to consider four phases for \( \Delta y_t \) at \( t \) depending on \( I_t^{(i)} \) and \( I_{t-1}^{(i)} \) \( (i = 1, 2) \). By taking the difference operation on (2.19) and using (2.17) and (2.18), we have the representation as

\[
(2.20) \quad \Delta y_t = \sigma(t) \left\{ A_0 + \sum_{i=1}^{p} A_i (t-i) \sigma(t-i)^{-1} \Delta y_{t-i} + \Delta u_t \right\},
\]

where \( A_0 = -\gamma_2^* \),

\[
A_1(t-1) = 1 - \gamma_1 \sigma(t-1),
\]

and

\[
A_i(t-i) = -\gamma_i \sigma(t-i) \quad (i = 2, \ldots, p).
\]

It is clear that the time series model defined by (2.4) is a special case of (2.20) when \( \{e_t\} \) are martingale difference sequences. Moreover, we have the following characterization result on \( \{\Delta y_t\} \).

**THEOREM 2.1.** Define the non-linear transformation of \( \{\Delta y_t\} \) by

\[
(2.21) \quad T(\Delta y_t) = \sigma(t)^{-1} \Delta y_t.
\]

Then the transformed stochastic process \( \{T(\Delta y_t)\} \) satisfies

\[
(2.22) \quad T(\Delta y_t) = A_0 + \sum_{i=1}^{p} A_i (t-i) T(\Delta y_{t-i}) + \Delta u_t,
\]

where \( A_0 \) and \( A_i(t-i) \) \( (i = 1, \ldots, p) \) are defined by (2.20).

The time series model defined by (2.22) could be called the threshold autoregressive (TAR) model with time-varying coefficients in the non-linear time series analysis. Because the transformed process \( T(\Delta y_t) \) is a TAR process, it is apparent that the original stochastic process for \( \{\Delta y_t\} \) is different from the TAR\((p)\) model. The next proposition gives an important difference between the SSAR\((p)\) models in this paper and the standard threshold AR (TAR) models in the non-linear time series analysis.

**THEOREM 2.2.** Suppose the density function \( f(w) \) of \( w_t \) \( (v_t = w_t \sqrt{T_t}) \) in the SSAR\((p)\)-ARCH\((r)\) model is everywhere positive with respect to the Lebesgue measure. Then given \( (\Delta y_{t-1}, \ldots, \Delta y_{t-p}, v_{t-1}, \ldots, v_{t-r}) = (z_1, \ldots, z_{p+r}) \), the conditional probability

\[
\Pr \{\Delta y_t \leq y \mid \Delta y_{t-1} = z_1, \ldots, \Delta y_{t-p} = z_p, v_{t-1} = z_{p+1}, \ldots, v_{t-r} = z_{p+r}\}
\]

is a continuous function of \( z' = (z_1, \ldots, z_{p+r}) \).
Contrary to the SSAR models discussed in this paper, it is well-known that the TAR models do not share this continuity property. Although this is an interesting aspect mathematically, but it may cause some difficulty in practical applications. (See Chapter 3 of Tong (1990), for instance.)

By using (2.20) with the coherency restrictions, the stochastic process \( \{ \Delta y_t \} \) can be further represented as

\[
(2.23) \Delta y_t = \begin{cases} 
  a_2^* + \sum_{i=1}^{p} a_i \Delta y_{t-i} + \sigma_1 \Delta u_t & \text{if } \Delta y_{t-1} \geq 0, \Delta y_t \geq 0 \\
  a_2^* + \left( \frac{\sigma_1}{\sigma_2} \right) b_1 \Delta y_{t-1} + \sum_{i=2}^{p} a_i \Delta y_{t-i} + \sigma_1 \Delta u_t & \text{if } \Delta y_{t-1} < 0, \Delta y_t \geq 0 \\
  b_2^* + \left( \frac{\sigma_2}{\sigma_1} \right) a_1 \Delta y_{t-1} + \sum_{i=2}^{p} b_i \Delta y_{t-i} + \sigma_2 \Delta u_t & \text{if } \Delta y_{t-1} \geq 0, \Delta y_t < 0 \\
  b_2^* + \sum_{i=1}^{r} b_i \Delta y_{t-i} + \sigma_2 \Delta u_t & \text{if } \Delta y_{t-1} < 0, \Delta y_t < 0
\end{cases}
\]

By this form of representation, we notice that the differenced process \( \{ \Delta y_t \} \) from the SSARI model not only has a simultaneous switching characteristic, but also a characteristic of the threshold type time series model. For the stochastic process \( \{ \Delta y_t \} \) defined by (2.23), we can present a set of sufficient conditions for its ergodicity. A proof is provided in the Appendix.

**Theorem 2.3.** Suppose (i) \( p \) and \( r \) are non-negative integers, (ii) the coherency condition (2.15) holds, (iii) \( \{ v_t \} \) are a sequence of martingale differences and the density function \( f(w) \) of \( w_t = w_t \sqrt{h_t} \) is positive everywhere with respect to the Lebesgue measure, \( E[|v_t^2|] < +\infty \), and the conditional variance function \( h_t = E[v_t^2 \mid \mathcal{F}_{t-1}] \) satisfy (2.7)–(2.8) (\( \alpha_i \geq 0 \)) with

\[
(2.24) \max \{ \sigma_1, \sigma_2 \} \sum_{i=1}^{r} \alpha_i < 1.
\]

Then the Markov chain defined by (2.23) for \( \{ \Delta y_t \} \) is geometrically ergodic if we have (iv)

\[
(2.25) \max \left\{ a_1 + \sum_{j=2}^{p} |a_j|, b_1 + \sum_{j=2}^{p} |b_j| \right\} < 1,
\]

\[
(2.26) \left[ a_1 - \sum_{j=2}^{p} |a_j| \right] \left[ b_1 - \sum_{j=2}^{p} |b_j| \right] < 1,
\]

and

\[
(2.27) \max \left\{ \sum_{j=2}^{p} |a_j|, \sum_{j=2}^{p} |b_j| \right\} < 1.
\]
For a precise definition and related discussions on the ergodicity and geometric ergodicity for Markov chains with the general state space, see Tweedie (1983), Tong (1990), or Meyn and Tweedie (1993). When \( p = 1 \) and \( r = 0 \), Kunitomo and Sato (1999) have shown that the necessary and sufficient conditions for the ergodicity of \( \{\Delta y_t\} \) are given by

\[
(2.28) \quad a_1 < 1, \quad b_1 < 1, \quad a_1 b_1 < 1.
\]

It is evident that the sufficient conditions in the above theorem are reduced to (2.28) when \( p = 1 \) and \( r = 0 \) only under the assumption \( E[|v_t|] < \infty \). Furthermore, if \( a_1 = b_1 \), then we have the standard condition \( |a_1| = |b_1| < 1 \) in the time series analysis. When \( p \geq 2 \) and \( r = 0 \), the naive sufficient conditions for the stability of the stochastic process \( \{\Delta y_t\} \) are given by

\[
(2.29) \quad \sum_{i=1}^{p} |a_i| < 1, \quad \sum_{i=1}^{p} |b_i| < 1.
\]

But these conditions preclude many interesting non-linear phenomena such as the case when there are some over-reactions and subsequent gradual adjustments to the mean level. It seems that the conditions we have for ergodicity in the above theorem cover some of these important cases, but still they are too restrictive as the necessary conditions when \( p \geq 2 \). From our limited number of simulations, we have found that the sufficient conditions we set exclude some important cases even when \( p = 2 \).

2.3. Estimation and Model Selection

In order to estimate the SSAR models we have proposed and investigated the maximum likelihood (ML) method (Kunitomo and Sato (1999, 2000)). We also propose using the ML method for estimating the SSARI models with ARCH disturbances. However, because of ARCH effects on the disturbance terms, we need to modify the likelihood function for the SSARI model slightly.

We set the initial conditions \( \Delta y_t(p + 1 \leq t \leq T) \) being fixed and \( v_t(t \leq 0) \) being zeros. Then, there is an important aspect in the present model that the Jacobian of the transformation from \( \{\Delta u_t, p + 1 \leq t \leq T\} \) to \( \{\Delta y_t, p + 1 \leq t \leq T\} \) is given by

\[
J = \prod_{t=p+1}^{T} \sigma(t)^{-1}.
\]

We assume that the disturbance terms \( \{w_t\} \) are a sequence of i.i.d. normal random variables followed by \( N(0, 1) \), where \( v_t = w_t \sqrt{h_t} \) as (2.6)–(2.8). By using (2.20) we have an alternative representation as \( v_t = f_t(\Delta y_t, \ldots, \Delta y_{t-p}, \theta) \) and

\[
(2.30) \quad f_t(\Delta y_t, \ldots, \Delta y_{t-p}, \theta) = \frac{\Delta y_t}{\sigma(t)} - A_0 - \sum_{i=1}^{p} \frac{A_i(t - i)}{\sigma(t - i)} \Delta y_{t-i},
\]
where \( \theta' = (\gamma^*, \gamma_1, \ldots, \gamma_p, \sigma_1, \sigma_2, \alpha_1, \ldots, \alpha_r) \) is a vector of structural parameters appearing in the SSARI(\( p \))-ARCH(\( r \)) model given by (2.20). Then the normalized log-likelihood function for \( \{\Delta y_t, p + 1 \leq t \leq T\} \) given the (fixed) initial conditions can be written as

\[
\log L_T(\theta) = -\frac{T - (p + 1)}{2T} \log 2\pi - \frac{1}{2T} \sum_{t=p+1}^{T} \sum_{i=1}^{2} I_{ti}^{(i)} \log[\sigma_i^2 h_i(\theta)] - \frac{1}{2T} \sum_{t=p+1}^{T} f_t(\theta)^2 \frac{h_t(\theta)}{h_t(\theta)},
\]

where we denote (2.30) as \( f_t(\theta) \) The maximum likelihood (ML) estimator can be defined as the maximum of \( \log L_T(\theta) \) with respect to the unknown parameters in \( \theta \), where the parameter space \( \Theta \) is restricted by the coherency conditions given by (2.15) and the ergodicity conditions imposed by Theorem 2.3. For an efficient ML computation, we can divide the full samples into samples corresponding to four different phases using (2.23). We then try to maximize the likelihood function with respect to the unknown parameters by an iterative optimization technique. This type of representation of the likelihood function and its maximization problem has been standard in nonlinear econometric models. (See Chapter 8 of Amemiya (1985), for instance.)

For the asymptotic properties of the ML estimator, we can establish that the ML estimator is consistent and asymptotically normal in the general SSARI(p)-ARCH(r) model. Because the proof is lengthy but standard (Basawa et al. (1976) and Weiss (1986) for instance), we have omitted the details.

**Theorem 2.4.** Let \( \{\Delta y_t\} \) be followed by a SSARI(p)-ARCH(r) model with \( \sigma_i > 0 \) (\( i = 1, 2 \)). Assume (i) a set of sufficient conditions for the coherency in (2.15) and the conditions for the geometrical ergodicity in Theorem 2.3, (ii) the disturbance terms \( \{v_t\} \) are a sequence of martingale differences and conditionally normally distributed given \( \mathcal{F}_{t-1} \) with \( E[v_t^{4+\delta}] < +\infty \) for some \( \delta > 0 \), (iii) \( E[(\Delta y_t)^2] < +\infty \). Also suppose (iv) the true parameter vector \( \theta_0 \) is an interior point in the parameter space \( \Theta \), which is a compact subspace of the Euclidean space. Then the ML estimator \( \hat{\theta}_{ML} \) of the unknown parameter vector in \( \theta \) is consistent and asymptotically normally distributed as

\[
\sqrt{T}(\hat{\theta}_{ML} - \theta_0) \overset{d}{\rightarrow} N(0, I(\theta_0)^{-1}),
\]

where \( \theta_0 \) is the true parameters vector and

\[
I(\theta_0) = \lim_{T \to \infty} \left[ -\frac{\partial^2 \log L_T(\theta)}{\partial \theta \partial \theta'} \right]_{\theta = \theta_0}.
\]

provided that \( I(\theta_0) \) is a positive definite matrix.

We need the existence of moments of the disturbances due to the ARCH terms and the second order moments of \( \Delta y_t \) in order to use the Ergodic Theorem
for the Markov chains. We have not as yet been able to obtain simple sufficient conditions for (iii) so that an alternative practical way is to modify the volatility function $h_t$ as

$$
(2.34) \quad h_t^* = 1 + \sum_{i=1}^{r} \alpha_i [v_{t-i}^2 I(v_{t-i}^2 < M) + MI(v_{t-i}^2 \geq M)] \quad a.s.,
$$

for a sufficiently large $M$ and the indicator function $I(\cdot)$. The abbreviated proof of the corresponding results without assuming (iii) when $p = 1$ and $r = 0$ has been given as Theorem 5 of Kunitomo and Sato (1999). The asymptotic information matrix can be consistently estimated by

$$
(2.35) \quad \left[ -\frac{\partial^2 \log L_T(\theta)}{\partial \theta \partial \theta'} \right]_{\hat{\theta}_{ML}}.
$$

Although we have obtained the consistency and the asymptotic normality of the ML estimator under a set of sufficient conditions, we do not presently know much about the behavior of the ML estimator when the sample size is not very large and the conditional distribution of disturbances are not normal. In this respect, we have investigated the finite sample properties of the ML estimator in a systematic way for the SSARI(1) model with ARCH(0) when $T = 100$ and $T = 500$ (Table 1 of Kunitomo and Sato (1999)). We have produced some evidence on the use of the ML estimation method for the SSARI models even when the sample size is not very large and the conditional distribution of disturbances are different from normal to a certain extent. From those results we expect that similar finite sample properties may hold in the present situation.

It is important to note that the class of the SSARI($p$)-ARCH($r$) models we have introduced in this paper includes many statistical time series models as special cases. For instance, it includes the integrated AR($p$) models with ARCH($r$) disturbances. In order to select an appropriate model within the class of the SSARI($p$)-ARCH($r$) models, we propose to use the minimum Akaike’s Information Criterion (AIC): see Akaike (1973). Although there have been many proposed criteria for model selection procedures, it is widely known that the minimum AIC is useful and successful in many applications. The value of AIC for the SSARI($p$)-ARCH($r$) model in the present case can be defined by

$$
(2.36) \quad \text{AIC}(p, r) = 2 \sum_{i=1}^{2} \sum_{t=p+1}^{T} I_t^{(i)} \log[\hat{\sigma}_i^2 h_t(\hat{\theta})] \\
+ \sum_{t=p+1}^{T} h_t(\hat{\theta})^{-1} f_t(\hat{\theta})^2 + 2(p + r),
$$

where $\hat{\sigma}_i$ ($i = 1, 2$) denote the ML estimator of $\sigma_i$ ($i = 1, 2$) and $\hat{\theta}$ denotes the vector of the ML estimator for $\theta$.

Furthermore, if we are interested in testing some restrictions on the unknown parameters of some SSARI($p$)-ARCH($r$) models in the classical hypothesis testing
framework, we can use this criterion. Let $AIC(p_1, r_1)$ and $AIC(p_2, r_2)$ be two AIC’s for two possible candidates in the class of the SSARI($p$)−ARCH($r$) models. Also let the likelihood ratio statistic be $LR(2 : 1)$ for testing the first model against the second model. Then we have the relation

$$AIC(p_1, r_1) - AIC(p_2, r_2) = 2 \log LR(2 : 1) + 2(k_1 - k_2), \quad (2.37)$$

where $k_i = p_i + r_i \ (i = 1, 2)$ are the number of coefficients. As is well-known, when we assume that the second model is included in the first model as a special case, the model selection by AIC corresponds to the likelihood ratio procedure with a significance level. Apparently, the choice between two alternative models by the minimum AIC criterion corresponds to a statistical testing problem on the underlying parameters in the class of the SSARI($p$)-ARCH($r$) models.

3. An application to financial data

In this section we shall report on an empirical result using some time series data on stock price indices. In our data analysis, we have used time series data of the Nikkei-225 indices which are the most popular stock price index traded in Japan and the SP-500 in the U.S. The first data sets are the closing daily data for the Nikkei-225 Spot index from January 1985 to December 1994, and the second data sets are the closing daily data of the SP-500 from January 1985 to December 1994. All data were transformed into their logarithms before the estimation of the non-stationary SSAR-ARCH model. It may be of some interest for economists to compare the time series movements of these spot prices, which are representative indices in two major stock markets in the world.

It has often been argued by financial economists and practitioners in the major financial markets that there are significant daily effects in the volatility functions of financial prices. The typical argument for the existence of daily effects is from the observation that the information flows are different from day to day and that there are some trading day effects and holiday effects. By taking account of this observation, in actual estimation we use the volatility function

$$h_t = 1 + \sum_{i=1}^r \alpha_i v_{t-i}^2 + \sum_{i=1}^{r^*} \beta_i D_{it}, \quad (3.1)$$

where $D_{it}$ are the dummy variables defined by

$$D_{it} = \begin{cases} 1 & \text{if } t \in J_i, \\ 0 & \text{if } t \notin J_i, \end{cases} \quad (3.2)$$

and the index sets $J_i \ (i = 1, \ldots, r^*)$ denote the calendar days. We took $r^* = 5$ in our actual investigation and $J_2$ corresponds to the Tuesday dummy, for instance. We could have incorporated other dummy variables such as holiday variables in a similar fashion.

We have estimated the SSARI($p$)-ARCH($r$) model with dummy variables of the form (3.1). The estimation of structural parameters in the SSARI($p$)-ARCH($r$) model has been conducted by the ML method under the assumption

<table>
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<th>ARCH(2)</th>
<th>ARCH(3)</th>
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</tr>
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<tr>
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Selected Model: SSARI(1)–ARCH(3) AIC: −4298.33

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<td></td>
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<td>–</td>
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<tr>
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<td>(3.61)</td>
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<td>0.60</td>
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<tr>
<td></td>
<td>(3.90)</td>
<td>(0.61)</td>
<td>(0.32)</td>
<td>(0.37)</td>
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Notes: The AIC values are calculated by (2.35) and (3.1). The figures in the parenthesis are the t-ratio, which is calculated by the estimates divided by the estimated standard deviation using (2.34). Also ARI(p) stands for the integrated p-th order AR model.

of conditionally normal disturbances. We have divided the full sample period for two sets of time series data into 4 sub-periods, each consisting of about 600 data points. This has been done because many different phenomena occurred in the full sample period and the assumption of constant coefficients in the underlying SSARI model could be unreasonable from a practical point of view. For instance, many changes in regulations, taxes, and market structures as well as financial innovations have occurred at financial markets in Japan during the full sample period. The values of AIC for various models in the class of the SSARI(p)-ARCH(r) models with dummy variables (0 ≤ p, r ≤ 3) and some of the estimation results are summarized in Table 1 for the Nikkei-225 data and in Table 2 for the SP-500 data. Because of space limitations, we only present the estimates of unknown coefficients in the selected models using the minimum AIC criterion with the dummy variables in the volatility function. The minimum AIC models among the selected models with and without the dummy variables are underlined for convenience. The numbers in the parentheses below the coefficient estimates are the corresponding t—ratios, which are calculated by using (2.23). Although we have estimated the volatility function without dummy variables, the estimated coefficients are similar in most cases and have been omitted. When the selected model is an ARI(p)-ARCH(r), the estimates of unknown coefficients are identical in the upward phase and the downward phase. Also it is straightforward to
Table 1 (cont.). Nikkei 225 Index (Feb. 21, 1987–Dec. 31, 1989).

<table>
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<th>ARCH(3)</th>
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Selected Model: SSARI(1)–ARCH(2) AIC: -5167.47

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<th>α₁</th>
<th>α₂</th>
<th>α₃</th>
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<td>-</td>
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<tr>
<td></td>
<td>(7.05)</td>
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<td>(8.24)</td>
<td>(3.29)</td>
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<tr>
<td>b (down)</td>
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<td>-</td>
<td>0.0068</td>
<td>(4.96)</td>
<td>(4.72)</td>
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Daily effects

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Table 1 (cont.). Nikkei 225 Index (Jan. 1, 1990–Jun. 11, 1992)

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Selected Model: ARI(2)–ARCH(3) AIC: -3225.92

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<td>(3.63)</td>
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Daily effects

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Table 1 (cont.)  Nikkei 225 Index (Jun. 12, 1992–Dec. 15, 1994)

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Selected Model: SSARI(2)–ARCH(3)  AIC: −3605.82

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<td>(2.80)</td>
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Daily effects  

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<tr>
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<td>−3639.58</td>
<td>−3641.81</td>
<td>−3639.80</td>
<td>−3638.09</td>
</tr>
<tr>
<td>ARI(1)</td>
<td>−3641.05</td>
<td>−3641.72</td>
<td>−3639.72</td>
<td>−3637.82</td>
</tr>
<tr>
<td>ARI(2)</td>
<td>−3641.15</td>
<td>−3642.10</td>
<td>−3639.91</td>
<td>−3638.20</td>
</tr>
<tr>
<td>ARI(3)</td>
<td>−3639.40</td>
<td>−3640.29</td>
<td>−3638.29</td>
<td>−3636.37</td>
</tr>
</tbody>
</table>

Selected Model: ARI(2)–ARCH(1)  AIC: −3642.10

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>σ</th>
<th>ARCH coef.</th>
<th>α_1</th>
<th>α_2</th>
<th>α_3</th>
</tr>
</thead>
<tbody>
<tr>
<td>AR coef.</td>
<td>0.0681</td>
<td>−0.0656</td>
<td>−</td>
<td>0.0085</td>
<td>ARCH coef.</td>
<td>0.04</td>
<td>−</td>
<td>−</td>
</tr>
<tr>
<td></td>
<td>(1.49)</td>
<td>(−1.54)</td>
<td>(15.03)</td>
<td></td>
<td>(1.11)</td>
<td></td>
<td></td>
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</table>

Daily effects  

<table>
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<tr>
<th>Mon.</th>
<th>Tue.</th>
<th>Wed.</th>
<th>Thu.</th>
<th>Fri.</th>
<th>Sat.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.95</td>
<td>0.68</td>
<td>1.00</td>
<td>0.62</td>
<td>−</td>
</tr>
<tr>
<td>(−0.47)</td>
<td>(−4.17)</td>
<td>(−0.04)</td>
<td>(−5.21)</td>
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</table>
Table 2(cont.). S & P 500 Index (Feb. 21, 1987–Dec. 31, 1989)

<table>
<thead>
<tr>
<th></th>
<th>AIC</th>
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<th>ARCH(1)</th>
<th>ARCH(2)</th>
<th>ARCH(3)</th>
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</thead>
<tbody>
<tr>
<td>SSARI(1)</td>
<td>−4188.95</td>
<td>−4360.52</td>
<td>−4413.92</td>
<td>−4440.44</td>
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</tr>
<tr>
<td>SSARI(2)</td>
<td>−4218.16</td>
<td>−4375.94</td>
<td>−4411.94</td>
<td>−4440.07</td>
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<tr>
<td>SSARI(3)</td>
<td>−4216.88</td>
<td>−4386.73</td>
<td>−4410.62</td>
<td>−4438.37</td>
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<tr>
<td>ARI(0)</td>
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<td>−4334.48</td>
<td>−4384.50</td>
<td>−4417.63</td>
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</tr>
<tr>
<td>ARI(1)</td>
<td>−4142.56</td>
<td>−4332.54</td>
<td>−4388.55</td>
<td>−4415.85</td>
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<tr>
<td>ARI(2)</td>
<td>−4176.52</td>
<td>−4354.78</td>
<td>−4387.23</td>
<td>−4414.74</td>
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</tr>
<tr>
<td>ARI(3)</td>
<td>−4177.53</td>
<td>−4365.64</td>
<td>−4388.12</td>
<td>−4414.06</td>
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</tr>
</tbody>
</table>

Selected Model: SSARI(1)–ARCH(3)  AIC: −4440.44

<table>
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<th>2</th>
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<th>σ</th>
<th>ARCH coef.</th>
<th>α1</th>
<th>α2</th>
<th>α3</th>
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</thead>
<tbody>
<tr>
<td>a (up)</td>
<td>0.1203</td>
<td>–</td>
<td>–</td>
<td>0.0069</td>
<td>ARCH coef.</td>
<td>0.32</td>
<td>0.08</td>
<td>0.13</td>
</tr>
<tr>
<td></td>
<td>(9.93)</td>
<td>(10.72)</td>
<td>(4.77)</td>
<td>(1.86)</td>
<td>(3.03)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>b (down)</td>
<td>−0.1812</td>
<td>–</td>
<td>–</td>
<td>0.0093</td>
<td>ARCH coef.</td>
<td>(4.77)</td>
<td>(1.86)</td>
<td>(3.03)</td>
</tr>
<tr>
<td></td>
<td>(8.04)</td>
<td>(7.00)</td>
<td>(7.00)</td>
<td>(7.00)</td>
<td>(7.00)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Mon. | Tue. | Wed. | Thu. | Fri. | Sat. |
Daily effects | 1   | 1.05 | 0.74 | 1.02 | 2.00 | – |
|        | (0.31) | (−3.68) | (0.18) | (3.25) | (3.25) |

calculate the likelihood ratio statistics by using (2.36) and the figures in tables. The results are in accordance with our discussions in this section.

In addition to the SSARI-ARCH models we also have estimated the ARI-EGARCH models and the SSARI-EGARCH models*. By using the Nikkei 225 data in the 1980’s we have found that the SSARI-EGARCH models are better than the ARI-EGARCH models with respect to the minimum AIC method. This may be because the SSARI models with conditional heteroskedasticities are better than the ARI models with conditional heteroskedasticities in the AIC sense. However, since the EGARCH models are more complicated than the ARCH models in our context, and there does not exist any statistical justification on their estimation (see the discussions in Page 356 of Nelson (1991)), we did not pursue the SSARI-EGARCH modelling in this study.

There are several interesting empirical observations from Table 1. First, the stock price indices sometimes show sharp asymmetrical movements in either the upward or downward phase. This phenomenon was evident both in 1985 and 1987. Actually we already knew that there was a sharp decline in October of 1987. During these sharp downward phases, the estimated values of the downward coefficients were often smaller than the corresponding upward coefficients. This agrees with the fact that the estimated volatility in the downward phase is often larger than the volatility in the upward phase. Second, we have found sig-

*We have done this analysis because the referee had suggested the possible use of the EGARCH models. We have estimated the EGARCH models with the maximum sum of orders being less than 3 under the Gaussian disturbance assumption.
nificant daily effects on the volatility functions in most cases. In our estimation of daily effects we have normalized the volatility coefficients such that the Monday volatility always takes the value of 1. Except for the first data period**, we have found smaller estimates of daily effects from the Monday volatility, which are statistically significant. This agrees with the notion that the Monday volatility effect is significantly larger than other days because of holiday effects, but it is less than the holiday length from Friday to Monday times the normal level of volatility.

In Japanese financial markets there was some trading on Saturday morning during 1980s, but this was terminated at the end of the 1980s. Also we found that the inclusion of daily effects in the form of (3.1) does not have a significant effect on the estimation of asymmetry or volatility level. Third, after around 1990 there were not many occasions (as was the case previously) when asymmetrical movements of price indices were evident and the estimated coefficients were not very large. The differences of AIC in ARI($p$) and SSARI($p$) are not very large in most cases in this period. There can be economic as well as institutional reasons for this phenomenon. For instance, we have compared the movements of the Nikkei-225 spot index and futures index (Kunitomo and Sato (1999)). The trading of the Nikkei-225 futures in Japan became active around at the end of 1989.

Fourth, as far as the SP-500 data is concerned, we find some evidence on the asymmetry discussed during the periods before around 1990. It has been widely known that this is quite evident for the data including that for 1987. However, in the data-period of 1990–1995, it is difficult to find some significant asymmetrical movements and the estimated coefficients are generally not as large as in Table 2. Fifth, in both markets it is often possible to find significant ARCH effects as well as daily effects on volatilities.

These empirical problems and findings may have some implications for financial economists. Needless to say, although our empirical results on stock indices are preliminary, it is not easy to detect the above features of the financial time series data we have mentioned with existing methods, with the linear time series modelling in particular.

4. Conclusions

In this paper we have focused on one important aspect of many financial economic time series which has been sometimes ignored in past econometric studies on the volatilities of major financial time series. Since the asymmetrical pattern in the movements of time series between the upward phase and the downward phase often observed by economists can not be represented properly by the standard ARMA, ARIMA, or ARCH processes, we have proposed to use a class

**The estimated Monday and Tuesday volatilities in this period are different from our intuition. However, Dr. Kunio Okina has suggested as one possible explanation that there was a substantial amount of information flow from the U.S. financial markets to the Tokyo stock market during the first period. Because there is a time lag between the two countries, the Tuesday volatility in Tokyo was quite large.
of nonstationary simultaneous switching autoregressive (SSAR) model with autoregressive conditional heteroskedastic (ARCH) disturbances. In this paper we have investigated some properties of the SSARI(p) -ARCH(r) model including the ergodicity in particular and the asymptotic properties of the maximum likelihood estimation method for estimating its unknown parameters. Unlike other methods already available for estimating the volatilities of financial time series, our method is a very simple but general one for investigating whether they are asymmetrical or not from a medium size set of time series data. The model selection procedure in the class of the SSARI(p)-ARCH(r) can be developed in a straightforward manner.

We have shown that there are some reasons why the SSARI model with ARCH disturbances is a useful tool to analyze many financial time series in financial markets. The argument is that if we permit the intrinsic value of security to be different from the observed price, and there exists a non-linear adjustment process, the result is the SSARI model. The estimated coefficients in the upward phase and in the downward phase can be different and the resulting volatility of financial time series can be asymmetrical around the normal volatility level.

By using an SSAR-ARCH model, we have examined the movements of the Nikkei-225 stock index in Tokyo and the SP-500 index in New York from 1985 to 1994. We have found some evidence of asymmetrical movements of stock indices in Japan before around 1990 in particular, which has been consistent with the view of some financial economists and practitioners in the Japanese financial markets. By using this empirical example, we have shown at least that our modelling approach is useful for analysing some financial prices.

Appendix

In this appendix, we give the proof of Theorem 2.3 in Section 2. We use the notation that \( x_t = \Delta y_t \) in the Appendix.

**Proof of Theorem 2.3.** Define a \( (p + r) \times 1 \) state vector \( X_t \) by

\[
X_t = \begin{pmatrix}
    x_t \\
    x_{t-1} \\
    \vdots \\
    x_{t-p+1} \\
    v_t^2 \\
    \vdots \\
    v_{t-r+1}^2
\end{pmatrix}.
\]

We then consider the Markovian representation for \( \{X_t\} \). The condition \( x_t \geq 0 \) is equivalent to \( w_t \sqrt{h_t} \geq \gamma_2^* + a_{t-1}'(x_{t-1}, \ldots, x_{t-p})' \), where

\[
a_{t-1}' = \left( \gamma_1 - \frac{1}{\sigma(t-1)}, \gamma_2, \ldots, \gamma_p \right).
\]
Since \( \sigma(t - 1) \) depends only on \( x_{t-1} \), the present phase conditions \( (I^1_t) \) and \( (I^2_t) \) depend on only the random variable \( w_t \) and finite past random variables in the state space vector. By using (2.20), we have the Markovian representation

\[
(A.3) \quad X_t = H(X_{t-1}, w_t),
\]

where

\[
H(X_{t-1}, w_t) = \begin{pmatrix}
-\sigma(t) \gamma^*_2 - \sigma(t) a'_{t-1} (x_{t-1}, \ldots, x_{t-p})' + \sigma(t) w_t \sqrt{h_t} \\
x_{t-1} \\
\vdots \\
x_{t-p+1} \\
w_t h_t \\
v^2_t h_t \\
\vdots \\
v^2_{t-r+1}
\end{pmatrix}.
\]

We use the criterion function

\[
(A.4) \quad G(x) = 1 + \max_{1 \leq j \leq p+r} |x(j)| \rho_j,
\]

for \( x = (x(i)) \) and some \( \rho_1 > \cdots > \rho_p > 0 \) and \( \rho_{p+1} > \cdots > \rho_{p+r} > 0 \). First, we consider the case when \( x_{t-1} \geq 0 \). For \( t \geq p + 1 \) and \( x = (x(j)) \),

\[
(A.5) \quad E \left[ \max_{1 \leq j \leq p} \rho_j |x_{t+1-j}| \mid X_{t-1} = x \right] 
\leq c_1 + \sigma \rho_1 \sum_{j=1}^{r} \alpha_j v^2_{t-j} + E \left[ \max \left\{ \left( a_1 \rho_1 x_{t-1} + \sum_{j=2}^{p} |a_j| \rho_1 |x_{t-j}| \right) I^1_t + \left( -\frac{\sigma_2}{\sigma_1} a_1 \rho_1 x_{t-1} + \sum_{j=2}^{p} |b_j| \rho_1 |x_{t-j}| \right) I^2_t, \max_{2 \leq j \leq p} \rho_j |x_{t+1-j}| \mid X_{t-1} = x \right] ,
\]

where \( c_1 \) is a positive constant, \( \sigma = \max\{\sigma_1, \sigma_2\} \) and we have the last inequality because \( h_t \geq 1 \) and \( \sqrt{h_t} \leq h_t \). Since we assumed the conditions (iv), there exist \( 0 \leq \theta_1 < 1 \) and \( \rho_1 > \rho_2 > \cdots > \rho_p > 0 \) such that \( \theta_1 > \rho_{j+1}/\rho_j \) \((j = 1, \ldots, p - 1)\) and

\[
(A.6) \quad \max \left\{ a_1 + \sum_{j=2}^{p} |a_j| \rho_1 \rho_j, b_1 + \sum_{j=2}^{p} |b_j| \rho_1 \rho_j \right\} < \theta_1 < 1,
\]

and
We note that the condition (2.26) in (iv) implies an inequality
\[(A.8)\]
\[\gamma_1 + \sum_{j=2}^{p} |\gamma_j| < \frac{1}{\sigma_1} + \frac{1}{\sigma_2}.\]

Hence we can take the same \(\theta_1\) in (A.6)–(A.7) satisfying \(0 \leq \theta_1 < 1\),
\[(A.9)\]
\[\max \left\{ \sigma_1 \left( \gamma_1 + \sum_{j=2}^{p} |\gamma_j| \frac{\rho_1}{\rho_j} - \frac{1}{\sigma_2} \right), \sigma_2 \left( \gamma_1 + \sum_{j=2}^{p} |\gamma_j| \frac{\rho_1}{\rho_j} - \frac{1}{\sigma_1} \right) \right\} < \theta_1 < 1.\]

Because of our conditions on the ARCH terms, there exist \(0 \leq \theta_2 < 1\) and \(\rho_{p+1} > \rho_{p+2} > \cdots > \rho_{p+r} > 0\) such that
\[(A.10)\]
\[\max \left\{ \sum_{j=1}^{r} \alpha_j \frac{\rho_{p+1}}{\rho_{p+j}}, \sigma \sum_{j=p}^{r} \alpha_j \frac{\rho_{p+1}}{\rho_{p+j}} \right\} < \theta_2 < 1.\]

Then we have
\[(A.11)\]
\[E[\rho_{p+1}v_t^2 | X_{t-1} = x] \leq c_2 + \rho_{p+1} \sum_{j=1}^{r} \alpha_j v_{t-j}^2 \]
\[< c_2 + \sum_{j=1}^{r} \left( \alpha_j \frac{\rho_{p+1}}{\rho_{p+j}} \right) \rho_{p+i}v_{t-j}^2 \]
\[\leq c_2 + \theta_2 \max_{1 \leq i \leq r} \{ \rho_{p+i}x_{(p+i)} \}.\]

Similarly, we have the relation \(\rho_1 \rho_{p+1} \sum_{j=1}^{r} \alpha_j v_{t-j}^2 < \theta_2 \max_{1 \leq i \leq r} \{ \rho_{p+i}v_{t-1}^2 \}\) in (A.5) by setting \(\rho_{p+1} = \rho_1 > 0\). Then by using (A.5), (A.9), and (A.11), we have
\[(A.12)\]
\[E[G(X_t) | X_{t-1} = x] < c_3 + \theta_3 G(x),\]
where \(\theta_3 = \max\{\theta_1, \theta_2\} < 1\) and \(c_3\) is a positive constant.

Next, we consider the case when \(x_{t-1} < 0\). By the similar arguments based on the conditions in (iv), we have a similar inequality for \(t \geq 1\) as (A.12) with \(\theta_4 (0 \leq \theta_4 < 1)\). Then for any \(X_{t-1} = x\), we have
\[(A.13)\]
\[E[G(X_t) | X_{t-1} = x] < c_4 + \theta_4 G(x),\]
where \(c_4\) is a positive constant.

The rest of our proof is similar to the one for Lemma 3.1 of Chan and Tong (1985). Since the Markov chain for \(X_t\) is aperiodic and \(\phi\)–irreducible due to our assumptions, we can apply Theorem 4 in Tweedie (1983) and establish that \(\{X_t\}\) is geometrically ergodic. \(Q.E.D.\)
Acknowledgements

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References