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THE EFFECTS OF NONNORMALITY ON THE UPPER PERCENTILES OF $T^2_{\max}$ STATISTIC IN ELLIPTICAL DISTRIBUTIONS

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In this paper, we consider the effects of nonnormality on the upper percentiles of $T^2_{\max}$ statistic in elliptical distributions. Some approximations based on the Bonferroni inequalities and asymptotic expansion procedure are given under the elliptical distribution setup. In order to achieve the purpose, asymptotic expansions for the distributions of univariate and bivariate Hotelling's T^2 type statistics are derived by a perturbation method when each population has the elliptical distribution. Finally, the accuracy of the approximations is investigated by Monte Carlo simulations for selected values of parameters.

Key words and phrases: Asymptotic expansion, Bonferroni inequality, Elliptical distribution, Monte Carlo simulation, Pairwise multiple comparison.

1. Introduction

Simultaneous confidence procedures for multiple comparisons among mean vectors have been considered by many authors. Simultaneous confidence intervals for pairwise multiple comparisons among mean vectors were first discussed by Roy and Bose (1953) based on Roy's union-intersection principle (see, e.g., Siotani, Hayakawa and Fujikoshi (1985, p. 227)). Also, Siotani (1959, 1960, 1964) proposed an approximate procedure to construct the simultaneous confidence intervals and discussed the approximate simultaneous confidence intervals in several important applications. Other important works are Fujikoshi and Seo (1999), Krishnaiah (1969, 1979), Krishnaiah and Reising (1985), Seo, Mano and Fujikoshi (1994), and so on. This paper is concerned with pairwise multiple comparisons among mean vectors under the elliptical populations. To construct the simultaneous confidence intervals, we need the upper percentiles of the $T^2_{\max}$ statistic when the populations have the elliptical distributions. For the case of pairwise comparisons under multivariate normal populations, this statistic is essentially reduced to the multivariate Studentized range statistic whose upper percentiles were discussed by Siotani (1959), Seo and Siotani (1992). In those papers, because of the difficulty in finding the exact upper percentiles of the statistic under multivariate normality, approximate upper percentiles of the statistic by the modified second order Bonferroni approximation procedure were given. This paper attempts to give the first order and the modified second order Bonferroni approximations for the upper percentiles of the statistic under the elliptical distribution setup. Further, we investigate the effects of nonnormality on upper percentiles of the

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multivariate Studentized range statistic in the elliptical distributions. Recently, the usual Hotelling's T^2 statistic in general distribution, i.e., under nonnormality is discussed by Kano (1995) and Fujikoshi (1997), but it is left as a future problem for the upper percentiles of $T^2_{\max}$ statistic in general distribution. The organization of the paper is as follows. In Section 2, an approximation to the upper percentiles of $T^2_{\max}$ statistic based on first order Bonferroni approximation procedure is discussed. In Section 3, as the modified approximation for the first approximation in Section 2, a modified second order Bonferroni approximation is described. Also, in order to evaluate this modified approximation, an asymptotic expansion for the distribution of bivariate Hotelling's T^2 type statistic in elliptical distributions is derived by a perturbation method. Finally, the accuracy of the approximations is investigated by Monte Carlo simulations for selected parameters in Section 4.

2. First order Bonferroni approximation

Let $\Pi_i (i = 1, \ldots, k)$ be the i-th population distributed as a p-dimensional elliptical distribution with mean vector $\boldsymbol{\mu}^{(i)}$ and covariance matrix $\boldsymbol{\Sigma}^{(i)} = \boldsymbol{\Sigma}$. Also, let $\boldsymbol{x}_1^{(i)}, \ldots, \boldsymbol{x}_{N_i}^{(i)}$ be $N_i (= N)$ independent sample vectors from $\Pi_i (i = 1, \ldots, k)$. Then, the i-th sample mean vector and the sample covariance matrix are

$$\bar{\boldsymbol{x}}^{(i)} = \frac{1}{N} \sum_{j=1}^{N} \boldsymbol{x}_j^{(i)},$$

$$\boldsymbol{S}^{(i)} = \frac{1}{N-1} \sum_{j=1}^{N} (\boldsymbol{x}_j^{(i)} - \bar{\boldsymbol{x}}^{(i)})(\boldsymbol{x}_j^{(i)} - \bar{\boldsymbol{x}}^{(i)})',$$

respectively. The usual simultaneous confidence intervals for pairwise comparisons among mean vectors are of the form

$$\boldsymbol{a}'(\boldsymbol{\mu}^{(\ell)} - \boldsymbol{\mu}^{(m)}) \in \left[\boldsymbol{a}'(\bar{\boldsymbol{x}}^{(\ell)} - \bar{\boldsymbol{x}}^{(m)}) \pm t\sqrt{2\boldsymbol{a}'\boldsymbol{S}\boldsymbol{a}/N} \right], \quad \forall \boldsymbol{a} \in \mathsf{R}^p - \{\boldsymbol{0}\}, \quad 1 \le \ell < m \le k, \tag{2.1}$$

where $\boldsymbol{S} = (1/k)\sum_{i=1}^{k} \boldsymbol{S}^{(i)}$, $\mathsf{R}^p - \{\boldsymbol{0}\}$ is the set of any nonnull real p-dimensional vectors, and t is some positive number. In order to obtain the simultaneous confidence intervals with the given confidence level $1 - \alpha$, it is necessary to give the value $t = t(p, k, N; \alpha) (> 0)$ such that

$$\Pr\{T^2_{\max} > t^2(p, k, N; \alpha)\} = \alpha,$$

where

$$T^2_{\max} = \max_{1 \le \ell < m \le k} \left\{ \frac{1}{2} (\boldsymbol{z}^{(\ell)} - \boldsymbol{z}^{(m)})' \boldsymbol{S}^{-1} (\boldsymbol{z}^{(\ell)} - \boldsymbol{z}^{(m)}) \right\}, \qquad \boldsymbol{z}^{(\ell)} = \sqrt{N}(\bar{\boldsymbol{x}}^{(\ell)} - \boldsymbol{\mu}^{(\ell)}).$$

In this paper, we discuss the upper percentiles of $T^2_{\max}$ in the elliptical distributions. An elliptical distribution is defined as $f(\boldsymbol{x}) = c_p|\boldsymbol{\Lambda}|^{-1/2} g((\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Lambda}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}))$, for some nonnegative function g, where c_p is the normalizing constant and

$\mathbf{\Lambda}$ is positive definite. For example, the multivariate normal, the multivariate t, the contaminated normal distributions belong to the class of elliptical distribution which is referred to e.g., by Muirhead (1982, p. 32), Fang, Kotz and Ng (1989). The characteristic function of the elliptical distribution is of the form $\phi(\boldsymbol{t}) = \exp(i\boldsymbol{t}'\boldsymbol{\mu})\psi(\boldsymbol{t}'\mathbf{\Lambda}\boldsymbol{t})$ for some function ψ, where $i = \sqrt{-1}$. Note that $\mathrm{E}[\boldsymbol{x}] = \boldsymbol{\mu}$ and $\mathrm{Cov}[\boldsymbol{x}] = \mathbf{\Sigma} = -2\psi'(0)\mathbf{\Lambda}$. We also define the kurtosis parameter by $\kappa = \{\psi''(0)/(\psi'(0))^2\} - 1$.

In general, it is difficult to obtain the exact upper percentiles of $T^2_{\max}$ even under multivariate normality. The approximate procedure based on the Bonferroni inequality is adopted in order to obtain a useful simultaneous confidence intervals estimation. By the first order Bonferroni inequality for $\Pr\{T^2_{\max} > t^2\}$;

$$\Pr\{T^2_{\max} > t^2\} < \sum\sum_{\ell<m} \Pr\{T^2_{\ell m} > t^2\},$$

where

$$T^2_{\ell m} = \frac{1}{2}(\boldsymbol{z}^{(\ell)} - \boldsymbol{z}^{(m)})'\boldsymbol{S}^{-1}(\boldsymbol{z}^{(\ell)} - \boldsymbol{z}^{(m)}),$$

the first order Bonferroni approximation $t_1^2(\alpha) = t_1^2$ is given as a critical value that satisfies the equality

$$\sum\sum_{\ell<m} \Pr\{T^2_{\ell m} > t_1^2\} = \alpha.$$

We note that t_1^2 is overestimated, and $T^2_{\ell m}$ is reduced to the Hotelling's T^2 type statistic(F statistic) when the populations have multivariate normal distributions. However, under the class of the elliptical distributions, $T^2_{\ell m}$ is no longer an F statistic. Also, this statistic is different from the Hotelling's T^2 statistic discussed by Iwashita (1997). In this section, therefore, we derive an asymptotic expansion for $T^2_{\ell m}$ by perturbation method. Note that

$$(N-1)\boldsymbol{S}^{(i)} = N\boldsymbol{W}^{(i)} - N(\bar{\boldsymbol{x}}^{(i)} - \boldsymbol{\mu}^{(i)})(\bar{\boldsymbol{x}}^{(i)} - \boldsymbol{\mu}^{(i)})',$$

where

$$\boldsymbol{W}^{(i)} = \frac{1}{N}\sum_{j=1}^{N}(\boldsymbol{x}_j^{(i)} - \boldsymbol{\mu}^{(i)})(\boldsymbol{x}_j^{(i)} - \boldsymbol{\mu}^{(i)})'.$$

Without loss of generality, we can assume $\mathbf{\Sigma} = \boldsymbol{I}$. Let

$$\bar{\boldsymbol{x}}^{(j)} = \boldsymbol{\mu}^{(j)} + \frac{1}{\sqrt{N}}\boldsymbol{z}^{(j)}, \qquad \boldsymbol{W}^{(j)} = \boldsymbol{I}_p + \frac{1}{\sqrt{N}}\boldsymbol{Z}^{(j)}$$

for $j = 1, \ldots, k$. Then we can write

$$\boldsymbol{S} = \frac{N}{N-1}\left(\boldsymbol{I}_p + \frac{1}{k\sqrt{N}}\sum_{i=1}^{k}\boldsymbol{Z}^{(i)} - \frac{1}{kN}\sum_{i=1}^{k}\boldsymbol{z}^{(i)}\boldsymbol{z}^{(i)\prime}\right),$$

and hence

$$\boldsymbol{S}^{-1} = \boldsymbol{I}_p - \frac{1}{\sqrt{N}}\boldsymbol{A}_0 + \frac{1}{N}\boldsymbol{A}_1 + \mathrm{o}_p(N^{-1}),$$

where

$$\boldsymbol{A}_0 = \frac{1}{k}\sum_{i=1}^{k}\boldsymbol{Z}^{(i)},$$

$$\boldsymbol{A}_1 = \frac{1}{k}\sum_{i=1}^{k}\boldsymbol{z}^{(i)}\boldsymbol{z}^{(i)\prime} + \frac{1}{k^2}\sum_{i=1}^{k}\sum_{j=1}^{k}\boldsymbol{Z}^{(i)}\boldsymbol{Z}^{(j)} - \boldsymbol{I}_p.$$

Note that $T^2_{\ell m} = \boldsymbol{\tau}'_{\ell m}\boldsymbol{S}^{-1}\boldsymbol{\tau}_{\ell m}$, where $\boldsymbol{\tau}_{\ell m} = (\boldsymbol{z}^{(\ell)} - \boldsymbol{z}^{(m)})/\sqrt{2}$. Therefore the characteristic function of $T^2_{\ell m}$ can be written as

$$E[\exp(itT^2_{\ell m})] = E\left[\exp\{it\boldsymbol{\tau}'_{\ell m}\boldsymbol{\tau}_{\ell m}\}\left[1 + \frac{1}{\sqrt{N}}(-it)\boldsymbol{\tau}'_{\ell m}\boldsymbol{A}_0\boldsymbol{\tau}_{\ell m} + \frac{1}{N}\left\{(it)\boldsymbol{\tau}'_{\ell m}\boldsymbol{A}_1\boldsymbol{\tau}_{\ell m} + \frac{1}{2}(it)^2(\boldsymbol{\tau}'_{\ell m}\boldsymbol{A}_0\boldsymbol{\tau}_{\ell m})^2\right\}\right]\right] + \mathrm{o}(N^{-1}).$$

Calculating the characteristic function of $T^2_{\ell m}$ with respect to $\boldsymbol{z}^{(j)}(=\sqrt{N}(\bar{\boldsymbol{x}}^{(j)} - \boldsymbol{\mu}^{(j)}))$ and $\boldsymbol{Z}^{(j)}(=\sqrt{N}(\boldsymbol{W}^{(j)} - \boldsymbol{I}_p))$ by using the joint density of $\boldsymbol{z}^{(j)}$ and $\boldsymbol{Z}^{(j)}$ given by Iwashita (1997), we obtain

$$E[e^{itT^2_{\ell m}}] = (1-2it)^{-p/2}\left\{1 + \frac{1}{4N}\{c^{(0)}_{\ell m} + c^{(1)}_{\ell m}(1-2it)^{-1} + c^{(2)}_{\ell m}(1-2it)^{-2}\}\right\} + \mathrm{o}(N^{-1}),$$

where

$$c^{(0)}_{\ell m} = -\frac{1}{k}p^2 + \frac{1}{2}p(p+2)\left\{\left(\frac{1}{4} - \frac{1}{k}\right)(\kappa_\ell + \kappa_m) - \frac{1}{k}\bar{\kappa}\right\}, \tag{2.2}$$

$$c^{(1)}_{\ell m} = -\frac{2}{k}p - p(p+2)\left\{\left(\frac{1}{4} - \frac{2}{k}\right)(\kappa_\ell + \kappa_m) + \frac{1}{k}\bar{\kappa}\right\}, \tag{2.3}$$

$$c^{(2)}_{\ell m} = \frac{1}{k}p(p+2) + \frac{1}{2}p(p+2)\left\{\left(\frac{1}{4} - \frac{3}{k}\right)(\kappa_\ell + \kappa_m) + \frac{3}{k}\bar{\kappa}\right\}, \tag{2.4}$$

and $\bar{\kappa} = (1/k)\sum_{j=1}^{k}\kappa_j$.

Therefore, inverting the characteristic function, we have the following theorem.

THEOREM 1. *The distribution of* $T^2_{\ell m} = (\boldsymbol{z}^{(\ell)} - \boldsymbol{z}^{(m)})'\boldsymbol{S}^{-1}(\boldsymbol{z}^{(\ell)} - \boldsymbol{z}^{(m)})/2$ *can be expanded as*

$$\Pr\{T^2_{\ell m} > t^2\} = \Pr\{\chi^2_p > t^2\} + \frac{1}{4N}\sum_{j=0}^{2}c^{(j)}_{\ell m}\Pr\{\chi^2_{p+2j} > t^2\} + o(N^{-1}),$$

and also its upper α *percentiles can be expanded as*

$$t^2_{\ell m}(\alpha) = \chi^2_p(\alpha) - \frac{1}{2N}\chi^2_p(\alpha)\left\{\frac{1}{p}c^{(0)}_{\ell m} - \frac{1}{p(p+2)}c^{(2)}_{\ell m}\chi^2_p(\alpha)\right\} + o(N^{-1}),$$

where the coefficients $c^{(0)}_{\ell m}$ *and* $c^{(2)}_{\ell m}$ *are given by* (2.2) *and* (2.4), *respectively, and* $\chi^2_p(\alpha)$ *is the upper* α *percentile of* χ^2 *distribution with* p *degrees of freedom.*

COROLLARY 2. *If the k populations have elliptical distributions with the same kurtosis parameter, i.e., $\kappa_j = \kappa$, $j = 1, \ldots, k$, then the coefficients of* (2.2), (2.3) *and* (2.4) *are given by*

$$c^{(0)}_{\ell m} = -\frac{1}{k}p^2 + \frac{1}{4k}p(p+2)(k-6)\kappa,$$
$$c^{(1)}_{\ell m} = -\frac{2}{k}p - \frac{1}{2k}p(p+2)(k-6)\kappa,$$
$$c^{(2)}_{\ell m} = \frac{1}{k}p(p+2) + \frac{1}{4k}p(p+2)(k-6)\kappa.$$

In addition, when $\kappa_j = 0$ for $j = 1, \ldots, k$, we can see that the coefficients $c^{(0)}_{\ell m}$, $c^{(1)}_{\ell m}$ and $c^{(2)}_{\ell m}$ coincide with the result of an asymptotic expansion for the upper percentile of the Hotelling's T^2-statistic under the normal assumption (see, e.g., Seo and Siotani (1992)). Note that $t^2_{\ell m} = (\nu p)F_{p,\nu-p+1}(\alpha)/(\nu - p + 1)$ under normality, where $\nu = (N-1)k$, and $F_{p,\nu-p+1}(\alpha)$ is the upper percentiles of F distribution with p and $\nu - p + 1$ degrees of freedoms. Therefore, we can write

$$t^2_{\ell m}(\alpha) = \frac{\nu p}{\nu - p + 1}F_{p,\nu-p+1}(\alpha) - \frac{1}{2N}\chi^2_p(\alpha)\left\{\left(\frac{1}{p}c^{(0)}_{\ell m} + \frac{1}{k}p\right) - \left(\frac{1}{p(p+2)}c^{(2)}_{\ell m} - \frac{1}{k}\right)\chi^2_p(\alpha)\right\} + \mathrm{o}(N^{-1}).$$

Also, it may be noted from Corollary 2 that, when $k = 6$, the coefficients $c^{(0)}_{\ell m}$, $c^{(1)}_{\ell m}$ and $c^{(2)}_{\ell m}$ do not depend on κ, that is, the asymptotic expansion for $k = 6$ coincides with an asymptotic expansion for the Hotelling's T^2-statistic under normality. If $\kappa_j = \kappa$, $j = 1, \ldots, k$, then the main effect of nonnormality is the following difference:

$$Q(p, k, N, \kappa; \alpha) = t^2_{\ell m}(\alpha) - \frac{\nu p}{\nu - p + 1}F_{p,\nu-p+1}(\alpha) = -\frac{1}{8kN}\chi^2_p(\alpha)\{p + 2 - \chi^2_p(\alpha)\}(k-6)\kappa.$$

Since $p + 2 < \chi^2_p(\alpha)$ for the cases of $\alpha = 0.1, 0.05, 0.01$ and $p \geq 2$, we note that when $k < 6$, $Q(= Q(p, k, N, \kappa; \alpha))$ is a negative value and decreasing function as $\kappa(> 0)$ is large.

For large N, an asymptotic expansion for t^2_1 is given by

$$t^2_1 = \chi^2_p(\alpha^*) - \frac{1}{2NM}\chi^2_p(\alpha^*)\sum\sum_{\ell<m}\left\{\frac{1}{p}c^{(0)}_{\ell m} - \frac{1}{p(p+2)}c^{(2)}_{\ell m}\chi^2_p(\alpha^*)\right\} + \mathrm{o}(N^{-1}),$$

where $\alpha^* = \alpha/M$ and $M = k(k-1)/2$, and as another expression, we can write

$$t^2_1 = \frac{\nu p}{\nu - p + 1}F_{p,\nu-p+1}(\alpha^*) - \frac{1}{2NM}\chi^2_p(\alpha^*)\left\{\left(\frac{1}{p}c^{(0)}_{\ell m} + \frac{1}{k}p\right) - \left(\frac{1}{p(p+2)}c^{(2)}_{\ell m} - \frac{1}{k}\right)\chi^2_p(\alpha^*)\right\} + \mathrm{o}(N^{-1}).$$

Thus, the approximate simultaneous confidence intervals for pairwise comparisons among mean vectors would be as in (2.1), except that the quantile t is replaced by t_1 which has two approximate values based on χ^2 and F approximations.

3. A modified second order Bonferroni approximation

In this section, a modified second order Bonferroni procedure is described to improve the first order Bonferroni approximation.

Let $\boldsymbol{y}_1 = \boldsymbol{z}^{(1)} - \boldsymbol{z}^{(2)}, \boldsymbol{y}_2 = \boldsymbol{z}^{(1)} - \boldsymbol{z}^{(3)}, \ldots, \boldsymbol{y}_M = \boldsymbol{z}^{(k-1)} - \boldsymbol{z}^{(k)}$. By the Bonferroni inequalities for $\Pr\{T^2_{\max} > t^2\}$, i.e.,

$$\sum_{i=1}^{M} \Pr\left\{\frac{1}{2}\boldsymbol{y}_i'\boldsymbol{S}^{-1}\boldsymbol{y}_i > t^2\right\} - \beta(t^2) < \Pr\{T^2_{\max} > t^2\} < \sum_{i=1}^{M} \Pr\left\{\frac{1}{2}\boldsymbol{y}_i'\boldsymbol{S}^{-1}\boldsymbol{y}_i > t^2\right\},$$

where

$$\beta(t^2) = \sum\sum_{i<j} \Pr\left\{\frac{1}{2}\boldsymbol{y}_i'\boldsymbol{S}^{-1}\boldsymbol{y}_i > t^2, \frac{1}{2}\boldsymbol{y}_j'\boldsymbol{S}^{-1}\boldsymbol{y}_j > t^2\right\},$$

the *modified second approximation* t^2_M by the modified second Bonferroni procedure is defined as a critical value that satisfies the equality

$$\sum_{i=1}^{M} \Pr\left\{\frac{1}{2}\boldsymbol{y}_i'\boldsymbol{S}^{-1}\boldsymbol{y}_i > t^2_M\right\} - \sum\sum_{i<j} \Pr\left\{\frac{1}{2}\boldsymbol{y}_i'\boldsymbol{S}^{-1}\boldsymbol{y}_i > t^2_1, \frac{1}{2}\boldsymbol{y}_j'\boldsymbol{S}^{-1}\boldsymbol{y}_j > t^2_1\right\} = \alpha.$$

In order to obtain the modified second approximation t^2_M, we discuss the evaluation of $\beta(t^2_1) = \sum\sum_{i<j} \Pr\{\frac{1}{2}\boldsymbol{y}_i'\boldsymbol{S}^{-1}\boldsymbol{y}_i > t^2_1, \frac{1}{2}\boldsymbol{y}_j'\boldsymbol{S}^{-1}\boldsymbol{y}_j > t^2_1\}$.

Consider two cases of joint probabilities to evaluate the $\beta(t^2_1)$; that is, $\beta_1(t^2_1) = \Pr\{T^2_{ij} > t^2_1, T^2_{k\ell} > t^2_1\}$ $(i \neq j \neq k \neq \ell)$ and $\beta_2(t^2_1) = \Pr\{T^2_{ij} > t^2_1, T^2_{ik} > t^2_1\}$ $(i \neq j \neq k)$ under the elliptical distribution setup. Note that the numbers of combinations for the types of $\beta_1(t^2_1)$ and $\beta_2(t^2_1)$ are given by $k(k-1)(k-2)(k-3)/8$ and $k(k-1)(k-2)/2$, respectively (see, Seo and Siotani (1992)). This implies that

$$\beta(t^2_1) = \frac{1}{8}k(k-1)(k-2)(k-3)\beta_1(t^2_1) + \frac{1}{2}k(k-1)(k-2)\beta_2(t^2_1)$$

when each population has the same distribution.

For large sample approximations, in this paper, asymptotic expansions of these joint probabilities can be obtained using the perturbation method. At first, we discuss an asymptotic expansion for $\Pr\{T^2_{ij} > t^2_1, T^2_{k\ell} > t^2_1\}$ $(i \neq j \neq k \neq \ell)$. For convenience, consider the joint characteristic function of T^2_{12} and T^2_{34}, that is,

$$C_1(it_1, it_2) = E[\exp(it_1 T^2_{12} + it_2 T^2_{34})].$$

Then, on the same line in Section 2, let

$$\bar{\boldsymbol{x}}^{(j)} = \boldsymbol{\mu}^{(j)} + \frac{1}{\sqrt{N}}\boldsymbol{z}^{(j)}, \qquad \boldsymbol{W}^{(j)} = \boldsymbol{I}_p + \frac{1}{\sqrt{N}}\boldsymbol{Z}^{(j)}$$

for $j = 1, 2, \dots, k$. The joint characteristic function $C_1(it_1, it_2)$ can be written as

$$C_1(it_1, it_2) = E[\exp(it_1 T_{12}^{(1)} + it_2 T_{34}^{(1)})] \left[1 + \frac{1}{\sqrt{N}} B_1 + \frac{1}{N} B_2\right] + \mathrm{o}(N^{-1}),$$

where

$$B_1 = it_1 T_{12}^{(2)} + it_2 T_{34}^{(2)}$$
$$B_2 = it_1 T_{12}^{(3)} + \frac{(it_1)^2}{2}(T_{12}^{(2)})^2 + it_2 T_{34}^{(3)} + \frac{(it_2)^2}{2}(T_{34}^{(2)})^2 + (it_1)(it_2) T_{12}^{(2)} T_{34}^{(2)},$$

and

$$T_{12}^{(1)} = \boldsymbol{\tau}'_{12}\boldsymbol{\tau}_{12}, \qquad T_{34}^{(1)} = \boldsymbol{\tau}'_{34}\boldsymbol{\tau}_{34},$$
$$T_{12}^{(2)} = -\boldsymbol{\tau}'_{12}\left(\frac{1}{k}\sum_{j=1}^{k}\boldsymbol{Z}^{(j)}\right)\boldsymbol{\tau}_{12}, \qquad T_{34}^{(2)} = -\boldsymbol{\tau}'_{34}\left(\frac{1}{k}\sum_{j=1}^{k}\boldsymbol{Z}^{(j)}\right)\boldsymbol{\tau}_{34},$$
$$T_{12}^{(3)} = \boldsymbol{\tau}'_{12}\left(\frac{1}{k}\sum_{j=1}^{k}\boldsymbol{z}^{(j)}\boldsymbol{z}^{(j)\prime} + \frac{1}{k^2}\sum_{i=1}^{k}\sum_{j=1}^{k}\boldsymbol{Z}^{(i)}\boldsymbol{Z}^{(j)} - \boldsymbol{I}_p\right)\boldsymbol{\tau}_{12},$$
$$T_{34}^{(3)} = \boldsymbol{\tau}'_{34}\left(\frac{1}{k}\sum_{j=1}^{k}\boldsymbol{z}^{(j)}\boldsymbol{z}^{(j)\prime} + \frac{1}{k^2}\sum_{i=1}^{k}\sum_{j=1}^{k}\boldsymbol{Z}^{(i)}\boldsymbol{Z}^{(j)} - \boldsymbol{I}_p\right)\boldsymbol{\tau}_{34},$$

and $\boldsymbol{\tau}_{12} = (\boldsymbol{z}^{(1)} - \boldsymbol{z}^{(2)})/\sqrt{2}$, $\boldsymbol{\tau}_{34} = (\boldsymbol{z}^{(3)} - \boldsymbol{z}^{(4)})/\sqrt{2}$.

Calculating the expectation $E[\exp(it_1 T_{12}^2 + it_2 T_{34}^2)]$ with respect to $\boldsymbol{z}^{(j)}$ and $\boldsymbol{Z}^{(j)}$, $j = 1, \dots, k$ by using the joint density of $\boldsymbol{z}^{(j)}$ and $\boldsymbol{Z}^{(j)}$, we have

$$\begin{aligned} E[\exp(it_1 T_{12}^2 + it_2 T_{34}^2)] \;=\;& (\lambda_1\lambda_2)^{-p/2} + \frac{1}{N}(\lambda_1\lambda_2)^{-p/2} \\ & \cdot\{a_1 + (a_{21}\lambda_1^{-1} + a_{22}\lambda_2^{-1}) + (a_{31}\lambda_1^{-2} + a_{32}\lambda_2^{-2}) \\ & \qquad + a_4(\lambda_1\lambda_2)^{-1}\} \\ & + \mathrm{o}(N^{-1}), \end{aligned}$$

where,

$$a_1 = -\frac{1}{2k}p(p-1) + \frac{1}{32k}(k-8)p(p+2)\sum_{j=1}^{4}\kappa_j,$$
$$a_{21} = -\frac{p}{k} + \frac{1}{16k}p(p+2)\left\{2\sum_{j=1}^{4}\kappa_j - (k-8)(\kappa_1+\kappa_2) - 8\bar{\kappa}\right\},$$
$$a_{22} = -\frac{p}{k} + \frac{1}{16k}p(p+2)\left\{2\sum_{j=1}^{4}\kappa_j - (k-8)(\kappa_3+\kappa_4) - 8\bar{\kappa}\right\},$$
$$a_{31} = \frac{1}{4k}p(p+2) + \frac{1}{32k}p(p+2)\{(k-12)(\kappa_1+\kappa_2) + 12\bar{\kappa}\},$$
$$a_{32} = \frac{1}{4k}p(p+2) + \frac{1}{32k}p(p+2)\{(k-12)(\kappa_3+\kappa_4) + 12\bar{\kappa}\},$$
$$a_4 = \frac{p}{2k} + \frac{1}{8k}p(p+2)\left\{-\sum_{j=1}^{4}\kappa_j + 2\bar{\kappa}\right\},$$

and $\lambda_1 = 1 - 2it_1$, $\lambda_2 = 1 - 2it_2$.

When $\kappa_j = \kappa$, we have

$$
\begin{aligned}
a_1 &= -\frac{1}{2k}p(p-1) + \frac{1}{8k}(k-8)p(p+2)\kappa, \\
a_{21} = a_{22} &= -\frac{p}{k} - \frac{1}{8k}p(p+2)(k-8)\kappa, \\
a_{31} = a_{32} &= \frac{1}{4k}p(p+2) + \frac{1}{16k}p(p+2)(k-6)\kappa, \\
a_4 &= \frac{p}{2k} - \frac{1}{4k}p(p+2)\kappa.
\end{aligned}
$$

Inverting this characteristic function $C_1(it_1, it_2)$, the following results are obtained.

Theorem 3. *For large N, an asymptotic expansion for the joint probability $\beta_1(t_1^2)$ is given by*

$$\Pr\{T_{ij}^2 > t_1^2, T_{k\ell}^2 > t_1^2\} = G_{p/2}^2(\eta) + \frac{1}{N}\{c_1 g_{p/2}(\eta)G_{p/2}(\eta) + c_2 g_{p/2}^2(\eta)\} + o(N^{-1}),$$

where

$$\eta = \frac{1}{2}t_1^2, \qquad G_{p/2}(\eta) = \int_\eta^\infty g_{p/2}(t)dt, \quad g_{p/2}(t) = \frac{1}{\Gamma\left(\frac{p}{2}\right)} t^{p/2-1}e^{-t},$$

and

$$
\begin{aligned}
c_1 = \frac{\eta}{16k}[16(p+2\eta) - \{(p+2)(k-4) - 2\eta(k-12)\}(\kappa_i + \kappa_j + \kappa_k + \kappa_\ell) \\
+8(p+6\eta+2)\bar{\kappa}],
\end{aligned}
$$

$$c_2 = \frac{\eta^2}{2kp}\{4 - (p+2)(\kappa_i + \kappa_j + \kappa_k + \kappa_\ell - 2\bar{\kappa})\}.$$

When $\kappa_j = \kappa$ for $j = 1, \ldots, k$, we have

$$
\begin{aligned}
c_1 &= \frac{\eta}{k}\left[p + 2\eta + \frac{1}{4}(k-6)\{-p+2\eta-2\}\kappa\right], \\
c_2 &= \frac{\eta^2}{kp}\{2 - (p+2)\kappa\}.
\end{aligned}
$$

Under normality, we note that the coefficients c_1 and c_2 are the same as the results of Seo (1995).

Secondly, consider $\Pr\{T_{ij}^2 > t_1^2, T_{ik}^2 > t_1^2\}(i \neq j \neq k)$. In this case, the joint characteristic function $C_2(it_1, it_2) = E[\exp(it_1 T_{12}^{(1)} + it_2 T_{13}^{(1)})]$ can be written as

$$C_2(it_1, it_2) = E[\exp(it_1 T_{12}^{(1)} + it_2 T_{13}^{(1)})]\left[1 + \frac{1}{\sqrt{N}}D_1 + \frac{1}{N}D_2\right] + \mathrm{o}(N^{-1}),$$

where

$$
\begin{aligned}
D_1 &= it_1 T_{12}^{(2)} + it_2 T_{13}^{(2)} \\
D_2 &= it_1 T_{12}^{(3)} + \frac{(it_1)^2}{2}(T_{12}^{(2)})^2 + it_2 T_{13}^{(3)} + \frac{(it_2)^2}{2}(T_{13}^{(2)})^2 + (it_1)(it_2)T_{12}^{(2)}T_{13}^{(2)},
\end{aligned}
$$

and

$$
\begin{aligned}
T_{12}^{(1)} &= \boldsymbol{\tau}_{12}'\boldsymbol{\tau}_{12}, \qquad T_{13}^{(1)} = \boldsymbol{\tau}_{13}'\boldsymbol{\tau}_{13},\\
T_{12}^{(2)} &= -\boldsymbol{\tau}_{12}'\left(\frac{1}{k}\sum_{j=1}^{k}\boldsymbol{Z}^{(j)}\right)\boldsymbol{\tau}_{12}, \qquad T_{13}^{(2)} = -\boldsymbol{\tau}_{13}'\left(\frac{1}{k}\sum_{j=1}^{k}\boldsymbol{Z}^{(j)}\right)\boldsymbol{\tau}_{13},\\
T_{12}^{(3)} &= \boldsymbol{\tau}_{12}'\left(\frac{1}{k}\sum_{j=1}^{k}\boldsymbol{z}^{(j)}\boldsymbol{z}^{(j)\prime} + \frac{1}{k^2}\sum_{i=1}^{k}\sum_{j=1}^{k}\boldsymbol{Z}^{(i)}\boldsymbol{Z}^{(j)} - \boldsymbol{I}_p\right)\boldsymbol{\tau}_{12},\\
T_{13}^{(3)} &= \boldsymbol{\tau}_{13}'\left(\frac{1}{k}\sum_{j=1}^{k}\boldsymbol{z}^{(j)}\boldsymbol{z}^{(j)\prime} + \frac{1}{k^2}\sum_{i=1}^{k}\sum_{j=1}^{k}\boldsymbol{Z}^{(i)}\boldsymbol{Z}^{(j)} - \boldsymbol{I}_p\right)\boldsymbol{\tau}_{13},
\end{aligned}
$$

and $\boldsymbol{\tau}_{12} = (\boldsymbol{z}^{(1)} - \boldsymbol{z}^{(2)})/\sqrt{2}$, $\boldsymbol{\tau}_{13} = (\boldsymbol{z}^{(1)} - \boldsymbol{z}^{(3)})/\sqrt{2}$.

Calculating the expectation $E[\exp(it_1T_{12}^2 + it_2T_{13}^2)]$ with respect to $\boldsymbol{z}^{(j)}$ and $\boldsymbol{Z}^{(j)}$, $j = 1, \ldots, k$ by using the joint density of $\boldsymbol{z}^{(j)}$ and $\boldsymbol{Z}^{(j)}$,

$$
C_2(it_1, it_2) = \phi^{-p/2}\left[1 + \frac{1}{N}\{b_1\phi^{-1} + b_2\phi^{-2}\}\right] + \mathrm{o}(N^{-1}),
$$

with the coefficients b_1 and b_2 given by

$$
\begin{aligned}
b_1 &= -\frac{p}{54k}\{36b_{11}(p+1) - 6b_{12}(p+2) - b_{13}k(p+2)\},\\
b_2 &= \frac{1}{162k}p(p+2)\{72b_{21} - 36b_{22} + 54b_{23} + b_{24}k\},
\end{aligned}
$$

where

$$
\begin{aligned}
b_{11} &= 3\lambda_1\lambda_2 - 2(\lambda_1 + \lambda_2) + 1,\\
b_{12} &= \{-4\lambda_1\lambda_2 + \lambda_1 + \lambda_2 + 2\}\kappa_1 + \{-4\lambda_1\lambda_2 + \lambda_1 - 2\lambda_2 + 5\}\kappa_2\\
&\quad + \{-4\lambda_1\lambda_2 - 2\lambda_1 + \lambda_2 + 5\}\kappa_3 + 3(\lambda_1 + \lambda_2 - 2)\bar{\kappa},\\
b_{13} &= 4(\lambda_1\lambda_2 - 1)\kappa_1 + \{4\lambda_1\lambda_2 - \lambda_1 - 2\lambda_2 - 1\}\kappa_2 + \{4\lambda_1\lambda_2 - 2\lambda_1 - \lambda_2 - 1\}\kappa_3,\\
b_{21} &= 4\lambda_1^2\lambda_2^2 - 4\lambda_1\lambda_2(\lambda_1 + \lambda_2) + (\lambda_1 + \lambda_2)^2,\\
b_{22} &= \{\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2(\lambda_1 + \lambda_2) + \lambda_1 + \lambda_2\}\kappa_1\\
&\quad + \{\lambda_1^2 + 4\lambda_2^2 - 2\lambda_1\lambda_2(\lambda_1 + 4\lambda_2) + 9\lambda_1\lambda_2 - 2(\lambda_1 + \lambda_2)\}\kappa_2\\
&\quad + \{4\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2(4\lambda_1 + \lambda_2) + 9\lambda_1\lambda_2 - 2(\lambda_1 + \lambda_2)\}\kappa_3,\\
b_{23} &= \{2(\lambda_1^2 + \lambda_2^2) - 4\lambda_1\lambda_2(\lambda_1 + \lambda_2) + 6\lambda_1\lambda_2 - (\lambda_1 + \lambda_2)\}\bar{\kappa},\\
b_{24} &= \{\lambda_1^2 + \lambda_2^2 - 8\lambda_1\lambda_2(\lambda_1 + \lambda_2) + 6\lambda_1\lambda_2 + 4(\lambda_1 + \lambda_2)\}\kappa_1\\
&\quad + \{\lambda_1^2 + 16\lambda_2^2 - 4\lambda_1\lambda_2(\lambda_1 + 6\lambda_2) + 24\lambda_1\lambda_2 - 3\lambda_1 - 10\lambda_2\}\kappa_2\\
&\quad + \{16\lambda_1^2 + \lambda_2^2 - 4\lambda_1\lambda_2(6\lambda_1 + \lambda_2) + 24\lambda_1\lambda_2 - 10\lambda_1 - 3\lambda_2\}\kappa_3,
\end{aligned}
$$

and $\lambda_1 = 1 - (3/2)it_1$ and $\lambda_2 = 1 - (3/2)it_2$, $\phi = (1 - 2it_1)(1 - 2it_2) - i^2t_1t_2$. We also note that

$$
\phi^{-p/2} = \left(\frac{3}{4}\right)^{p/2}\sum_{m=0}^{\infty}\frac{\left(\frac{1}{2}p\right)_m}{m!}\cdot\frac{\left(\frac{1}{4}\right)^m}{\lambda_1^{p/2+m}\lambda_2^{p/2+m}},
$$

where

$$\left(\frac{1}{2}p\right)_m = \frac{p}{2}\left(\frac{p}{2}+1\right)\cdots\left(\frac{p}{2}+m-1\right).$$

Therefore, we have the following theorem.

THEOREM 4. *For large N, an asymptotic expansion for the joint probability $\beta_2(t_1^2)$ is given by*

$$\begin{aligned}
&\Pr\{T_{ij}^2 > t_1^2, T_{ik}^2 > t_1^2\} \\
&\quad = \left(\frac{3}{4}\right)^{p/2} \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}p\right)_m}{m!} \left(\frac{1}{4}\right)^m \\
&\qquad \cdot \left[G_{p/2+m}^2(\eta) + \frac{1}{N}\{d_1 g_{p/2+m}(\eta) G_{p/2+m}(\eta) + d_2 g_{p/2+m}^2(\eta)\} + o(N^{-1})\right],
\end{aligned}$$

where

$$\eta = \frac{2}{3}t_1^2,\ \ G_{p/2+m}(\eta) = \int_\eta^\infty g_{p/2+m}(t)dt,\ \ g_{p/2+m}(t) = \frac{1}{\Gamma\left(\frac{p}{2}+m\right)} t^{p/2+m-1}e^{-t},$$

and

$$\begin{aligned}
d_1 &= \frac{\eta}{144k}[144(2\eta + p - 2m) + 36d_{11} + d_{12}k], \\
d_2 &= \frac{\eta^2}{72k(p+2m)}[144(2m+1) + 12d_{21} + d_{22}k], \\
d_{11} &= -2(4\eta - p + 2m - 2)\kappa_i - (2\eta - p + 10m - 2)(\kappa_j + \kappa_k) \\
&\quad + 2(3\eta + p + 6m + 2)\bar{\kappa}, \\
d_{12} &= 2(10\eta - 9p + 14m - 18)\kappa_i + (8\eta - 9p + 22m - 18)(\kappa_j + \kappa_k), \\
d_{21} &= -2(6\eta + p + 6m + 2)\kappa_i + (24\eta - 5p - 30m - 10)(\kappa_j + \kappa_k) \\
&\quad - 6(3\eta - p - 6m - 2)\bar{\kappa} \\
d_{22} &= 2(8\eta - p + 14m - 2)\kappa_i - (26\eta - 7p - 22m - 14)(\kappa_j + \kappa_k).
\end{aligned}$$

When $\kappa_j = \kappa$ for $j = 1, \ldots, k$, we have

$$\begin{aligned}
d_1 &= \frac{\eta}{k}(2\eta + p - 2m) + \frac{\eta}{4k}(k-6)(\eta - p + 2m - 2)\kappa, \\
d_2 &= \frac{2\eta^2}{k(p+2m)}(2m+1) - \frac{\eta^2}{6k(p+2m)}(k-6)(3\eta - p - 6m - 2)\kappa.
\end{aligned}$$

Under normality, we also note that the coefficients d_1 and d_2 are the same as the results of Seo (1995). As a remark, since the asymptotic expansion for the joint characteristic function $C_2(it_1, it_2)$ can be represented by some expressions, it may be noted that there exist some expressions for the asymptotic expansion for the joint probability $\beta_2(t_1^2)$.

Thus, the approximate simultaneous confidence intervals for pairwise comparisons among mean vectors would be as in (2.1), except the quantile t is replaced by t_M, which can be obtained by an asymptotic expansion for $\beta(t_1^2)$.

4. Accuracy of approximation

In order to examine accuracy of the obtained approximations, we give simulation results of the upper percentiles of $T_{\max}$ statistic for selected values of parameters.

The Monte Carlo simulation was based on 100 replications of 10,000 simulations for selected values of p, k, N, κ, and α. Then the simulated value of the upper percentile of $T_{\max}$ is defined by the average of those 100 estimated values.

Tables 1 to 6 give the values of first order approximation based on χ^2 distribution ($t_{1\cdot\chi}$), the one based on F distribution ($t_{1\cdot F}$), the modified second-order approximation based on χ^2 distribution ($t_{M\cdot\chi}$), and the one based on F distribution ($t_{M\cdot F}$). The simulated value(t^*) is also given in the Tables. Each value is calculated for the following combinations of parameter values: $p = 2, 3, 5$,

Table 1. Approximate and simulated values for the case of $\alpha = 0.10$.

Multivariate normal distribution($\kappa = 0$)										
$p = 2$										
k	N	$t_{1\cdot\chi}$	$t_{1\cdot F}$	$t_{M\cdot\chi}$	$t_{M\cdot F}$	$\widehat{t_{1\cdot\chi}}$	$\widehat{t_{1\cdot F}}$	$\widehat{t_{M\cdot\chi}}$	$\widehat{t_{M\cdot F}}$	t^*
3	10	2.793	2.842	2.730	2.739	2.807	2.856	2.743	2.796	2.745
	20	2.702	2.713	2.643	2.645	2.706	2.717	2.647	2.659	2.639
	40	2.656	2.658	2.600	2.600	2.657	2.659	2.601	2.604	2.587
6	10	3.320	3.353	3.221	3.233	3.320	3.353	3.209	3.251	3.213
	20	3.244	3.251	3.148	3.151	3.244	3.251	3.139	3.149	3.133
	40	3.205	3.207	3.113	3.113	3.205	3.207	3.108	3.110	3.098
10	10	3.618	3.639	3.508	3.517	3.594	3.616	3.463	3.494	3.495
	20	3.557	3.562	3.450	3.452	3.551	3.556	3.428	3.435	3.436
	40	3.526	3.528	3.423	3.423	3.525	3.526	3.411	3.413	3.411
$p = 3$										
3	10	3.227	3.318	3.173	3.188	3.249	3.339	3.193	3.294	3.223
	20	3.093	3.113	3.039	3.042	3.099	3.119	3.044	3.067	3.040
	40	3.023	3.028	2.971	2.972	3.025	3.030	2.973	2.978	2.963
6	10	3.711	3.762	3.628	3.644	3.711	3.762	3.616	3.683	3.632
	20	3.605	3.617	3.519	3.523	3.605	3.617	3.509	3.524	3.508
	40	3.551	3.554	3.467	3.468	3.551	3.554	3.460	3.464	3.453
10	10	3.981	4.014	3.888	3.901	3.952	3.984	3.834	3.880	3.882
	20	3.900	3.908	3.805	3.808	3.892	3.899	3.778	3.788	3.793
	40	3.859	3.861	3.766	3.767	3.857	3.859	3.751	3.753	3.756
$p = 5$										
3	10	3.945	4.165	3.910	3.930	3.978	4.196	3.942	4.179	4.062
	20	3.719	3.767	3.674	3.681	3.729	3.777	3.683	3.737	3.697
	40	3.602	3.613	3.554	3.556	3.604	3.615	3.557	3.569	3.551
6	10	4.345	4.449	4.289	4.310	4.345	4.449	4.280	4.406	4.317
	20	4.180	4.203	4.109	4.116	4.180	4.203	4.098	4.128	4.101
	40	4.095	4.100	4.021	4.023	4.095	4.100	4.012	4.020	4.012
10	10	4.568	4.627	4.501	4.517	4.528	4.587	4.433	4.514	4.504
	20	4.446	4.459	4.367	4.372	4.435	4.448	4.332	4.351	4.358
	40	4.383	4.387	4.303	4.304	4.380	4.384	4.282	4.287	4.294

Table 2. Continued.

		Elliptical distribution(Contaminated normal;$\kappa = 1.78$)								
$p = 2$										
k	N	$t_{1\cdot\chi}$	$t_{1\cdot F}$	$t_{M\cdot\chi}$	$t_{M\cdot F}$	$\widehat{t_{1\cdot\chi}}$	$\widehat{t_{1\cdot F}}$	$\widehat{t_{M\cdot\chi}}$	$\widehat{t_{M\cdot F}}$	t^*
	10	2.716	2.766	2.629	2.642	2.804	2.852	2.740	2.793	2.693
3	20	2.663	2.674	2.590	2.592	2.699	2.710	2.640	2.654	2.610
	40	2.635	2.638	2.572	2.573	2.649	2.651	2.593	2.598	2.573
	10	3.320	3.353	3.222	3.233	3.320	3.353	3.209	3.251	3.206
6	20	3.244	3.251	3.148	3.150	3.244	3.251	3.139	3.149	3.135
	40	3.205	3.207	3.112	3.113	3.205	3.207	3.108	3.110	3.099
	10	3.739	3.760	3.672	3.678	3.600	3.622	3.468	3.499	3.558
10	20	3.619	3.624	3.536	3.538	3.562	3.567	3.435	3.443	3.476
	40	3.558	3.559	3.467	3.467	3.537	3.538	3.418	3.419	3.433
$p = 3$										
	10	3.114	3.208	3.035	3.057	3.245	3.336	3.190	3.291	3.156
3	20	3.034	3.055	2.965	2.969	3.091	3.111	3.037	3.060	3.001
	40	2.993	2.998	2.933	2.934	3.015	3.020	2.963	2.970	2.942
	10	3.711	3.762	3.632	3.647	3.711	3.762	3.616	3.683	3.621
6	20	3.605	3.617	3.521	3.525	3.605	3.617	3.509	3.525	3.508
	40	3.551	3.554	3.468	3.469	3.551	3.554	3.460	3.464	3.456
	10	4.134	4.165	4.088	4.095	3.957	3.989	3.838	3.885	3.954
10	20	3.979	3.986	3.913	3.915	3.903	3.911	3.786	3.797	3.843
	40	3.899	3.901	3.822	3.822	3.870	3.872	3.759	3.762	3.785
$p = 5$										
	10	3.766	3.996	3.707	3.743	3.976	4.195	3.941	4.178	3.975
3	20	3.626	3.675	3.564	3.574	3.721	3.769	3.676	3.729	3.636
	40	3.554	3.565	3.496	3.499	3.592	3.603	3.546	3.559	3.519
	10	4.345	4.449	4.295	4.315	4.345	4.449	4.280	4.406	4.299
6	20	4.180	4.203	4.114	4.121	4.180	4.203	4.098	4.128	4.097
	40	4.095	4.100	4.025	4.026	4.095	4.100	4.013	4.020	4.013
	10	4.776	4.831	4.753	4.759	4.530	4.589	4.436	4.516	4.587
10	20	4.553	4.567	4.508	4.511	4.444	4.457	4.340	4.359	4.422
	40	4.438	4.441	4.378	4.379	4.395	4.398	4.293	4.297	4.332

$k = 3, 6, 10$, $N_j(= N) = 20, 40, 60$ and $\alpha = 0.1, 0.05, 0.01$. For the distributions of each population, the multivariate normal($\kappa = 0$) and the contaminated normal($\varepsilon = 0.1, \sigma = 3 : \kappa = 1.78$) are treated, where the populations for each parameter set $(p, k, N; \alpha)$ have the same distribution. In practical use, we must estimate the kurtosis parameter since κ is unknown. Note here that an estimate of κ, which is a proper consistent estimate, is known as the estimation by Mardia's sample measure of multivariate kurtosis and we can estimate κ as follows. Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ be $p \times 1$ sample vectors from an elliptical distribution with mean vector $\boldsymbol{\mu}$, covariance matrix $\boldsymbol{\Sigma}$, and kurtosis parameter κ. Then,

$$E[\{(\boldsymbol{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x} - \boldsymbol{\mu})\}^2] = p(p + 2)(\kappa + 1).$$

Table 3. Approximate and simulated values for the case of $\alpha = 0.05$.

Multivariate normal distribution($\kappa = 0$)										
$p = 2$										
k	N	$t_{1\cdot\chi}$	$t_{1\cdot F}$	$t_{M\cdot\chi}$	$t_{M\cdot F}$	$\widehat{t}_{1\cdot\chi}$	$\widehat{t}_{1\cdot F}$	$\widehat{t}_{M\cdot\chi}$	$\widehat{t}_{M\cdot F}$	t^*
	10	3.095	3.162	3.053	3.062	3.118	3.184	3.075	3.147	3.092
3	20	2.981	2.996	2.940	2.942	2.987	3.003	2.946	2.962	2.945
	40	2.922	2.925	2.883	2.883	2.923	2.927	2.884	2.888	2.877
	10	3.561	3.601	3.492	3.503	3.561	3.601	3.486	3.535	3.500
6	20	3.471	3.480	3.402	3.404	3.471	3.480	3.397	3.408	3.398
	40	3.424	3.427	3.358	3.359	3.424	3.427	3.355	3.358	3.353
	10	3.830	3.855	3.751	3.759	3.801	3.827	3.710	3.745	3.752
10	20	3.760	3.766	3.681	3.683	3.752	3.758	3.665	3.673	3.676
	40	3.724	3.726	3.648	3.649	3.722	3.724	3.640	3.642	3.645
$p = 3$										
	10	3.535	3.654	3.502	3.514	3.567	3.685	3.533	3.662	3.587
3	20	3.371	3.398	3.335	3.338	3.380	3.407	3.344	3.373	3.345
	40	3.286	3.293	3.251	3.252	3.289	3.295	3.253	3.260	3.248
	10	3.952	4.014	3.897	3.910	3.952	4.014	3.891	3.967	3.920
6	20	3.829	3.843	3.769	3.773	3.829	3.843	3.764	3.781	3.769
	40	3.766	3.770	3.707	3.708	3.766	3.770	3.703	3.707	3.700
	10	4.192	4.230	4.128	4.138	4.156	4.194	4.078	4.128	4.137
10	20	4.100	4.108	4.032	4.034	4.090	4.098	4.011	4.023	4.026
	40	4.053	4.055	3.986	3.986	4.050	4.052	3.976	3.978	3.982
$p = 5$										
	10	4.264	4.540	4.247	4.259	4.311	4.584	4.294	4.579	4.465
3	20	4.001	4.061	3.974	3.980	4.015	4.075	3.987	4.052	4.013
	40	3.863	3.877	3.833	3.834	3.867	3.881	3.836	3.852	3.838
	10	4.589	4.711	4.556	4.571	4.589	4.711	4.553	4.690	4.616
6	20	4.403	4.430	4.356	4.362	4.403	4.430	4.351	4.384	4.358
	40	4.307	4.313	4.256	4.258	4.307	4.313	4.251	4.260	4.254
	10	4.778	4.845	4.735	4.748	4.731	4.798	4.673	4.757	4.759
10	20	4.642	4.658	4.588	4.592	4.629	4.645	4.562	4.582	4.588
	40	4.573	4.577	4.516	4.517	4.569	4.573	4.503	4.508	4.514

Therefore, a consistent estimator of the kurtosis parameter is given by

$$\widehat{\kappa} = \frac{1}{p(p+2)} b_{2,p} - 1,$$

where

$$b_{2,p} = \frac{1}{N} \sum_{i=1}^{N} \{(\boldsymbol{x}_i - \bar{\boldsymbol{x}})' \boldsymbol{S}^{-1} (\boldsymbol{x}_i - \bar{\boldsymbol{x}})\}^2.$$

It may be noted that more efficient estimate could be found. In this paper, however, we present the approximations based on Mardia's estimate $\widehat{\kappa}$ instead of κ. The estimation of kurtosis parameter in elliptical distributions is discussed by Seo and Toyama (1996).

Table 4. Continued.

Elliptical distribution(Contaminated normal;$\kappa = 1.78$)										
$p = 2$										
k	N	$t_{1\cdot\chi}$	$t_{1\cdot F}$	$t_{M\cdot\chi}$	$t_{M\cdot F}$	$\widehat{t_{1\cdot\chi}}$	$\widehat{t_{1\cdot F}}$	$\widehat{t_{M\cdot\chi}}$	$\widehat{t_{M\cdot F}}$	t^*
	10	2.969	3.038	2.905	2.919	3.113	3.179	3.070	3.142	3.010
3	20	2.916	2.931	2.862	2.865	2.976	2.991	2.935	2.952	2.896
	40	2.889	2.893	2.843	2.844	2.911	2.914	2.872	2.877	2.854
	10	3.561	3.601	3.495	3.505	3.561	3.601	3.486	3.535	3.486
6	20	3.471	3.480	3.403	3.405	3.471	3.480	3.397	3.409	3.396
	40	3.424	3.427	3.358	3.359	3.424	3.427	3.355	3.358	3.351
	10	3.979	4.003	3.938	3.943	3.808	3.834	3.717	3.751	3.823
10	20	3.836	3.842	3.781	3.782	3.766	3.772	3.676	3.684	3.725
	40	3.763	3.765	3.699	3.700	3.737	3.739	3.651	3.653	3.671
$p = 3$										
	10	3.362	3.487	3.306	3.327	3.561	3.680	3.528	3.657	3.490
3	20	3.282	3.309	3.232	3.237	3.368	3.395	3.332	3.361	3.290
	40	3.241	3.247	3.198	3.199	3.273	3.280	3.239	3.246	3.214
	10	3.952	4.014	3.901	3.914	3.952	4.014	3.891	3.967	3.899
6	20	3.829	3.843	3.772	3.775	3.829	3.843	3.764	3.782	3.762
	40	3.766	3.770	3.708	3.709	3.766	3.770	3.703	3.708	3.700
	10	4.376	4.412	4.350	4.355	4.162	4.200	4.084	4.134	4.219
10	20	4.195	4.203	4.152	4.154	4.103	4.112	4.023	4.035	4.089
	40	4.101	4.103	4.049	4.049	4.066	4.069	3.989	3.991	4.019
$p = 5$										
	10	4.010	4.302	3.971	3.999	4.309	4.582	4.292	4.576	4.355
3	20	3.868	3.929	3.825	3.834	4.004	4.064	3.976	4.041	3.931
	40	3.795	3.809	3.755	3.757	3.850	3.864	3.820	3.835	3.788
	10	4.589	4.711	4.561	4.574	4.589	4.711	4.553	4.690	4.586
6	20	4.403	4.430	4.361	4.366	4.403	4.430	4.351	4.384	4.350
	40	4.307	4.313	4.259	4.261	4.307	4.313	4.252	4.260	4.251
	10	5.023	5.086	5.012	5.015	4.733	4.801	4.676	4.759	4.852
10	20	4.770	4.785	4.743	4.745	4.640	4.655	4.573	4.593	4.660
	40	4.638	4.642	4.599	4.600	4.586	4.590	4.518	4.523	4.557

On the results based on $\widehat{\kappa}$ in Tables, we first calculate $\widehat{\kappa}$ by simulation for sample size kN under the elliptical distribution, and calculate the approximations by using $\widehat{\kappa}$ instead of κ. This process is repeated 100 times to obtain 100 approximations, and the average for 100 approximations is given in Tables. We denote the approximations $t_{1\cdot\chi}$, $t_{1\cdot F}$, $t_{M\cdot\chi}$, and $t_{M\cdot F}$ based on $\widehat{\kappa}$ by $\widehat{t_{1\cdot\chi}}$, $\widehat{t_{1\cdot F}}$, $\widehat{t_{M\cdot\chi}}$, and $\widehat{t_{M\cdot F}}$, respectively. Figures 1 and 2 show the relative errors of these approximations, i.e.,

$$FAC(\kappa) \equiv FA(p, k, N, t_{1\cdot\chi}; \alpha) = \frac{t_{1\cdot\chi} - t^*}{t^*},$$

$$MSAC(\kappa) \equiv MSA(p, k, N, t_{M\cdot\chi}; \alpha) = \frac{t_{M\cdot\chi} - t^*}{t^*},$$

Table 5. Approximate and simulated values for the case of $\alpha = 0.01$.

Multivariate normal distribution($\kappa = 0$)										
$p = 2$										
k	N	$t_{1\cdot\chi}$	$t_{1\cdot F}$	$t_{M\cdot\chi}$	$t_{M\cdot F}$	$\widehat{t}_{1\cdot\chi}$	$\widehat{t}_{1\cdot F}$	$\widehat{t}_{M\cdot\chi}$	$\widehat{t}_{M\cdot F}$	t^*
	10	3.736	3.856	3.721	3.727	3.783	3.902	3.769	3.893	3.819
3	20	3.561	3.589	3.543	3.545	3.575	3.602	3.557	3.586	3.558
	40	3.471	3.477	3.453	3.453	3.474	3.481	3.456	3.463	3.456
	10	4.081	4.143	4.052	4.060	4.081	4.143	4.051	4.120	4.087
6	20	3.955	3.969	3.922	3.925	3.955	3.969	3.921	3.938	3.923
	40	3.890	3.894	3.858	3.859	3.890	3.894	3.857	3.861	3.857
	10	4.290	4.327	4.254	4.261	4.248	4.285	4.208	4.252	4.275
10	20	4.197	4.206	4.158	4.160	4.185	4.194	4.145	4.155	4.159
	40	4.150	4.152	4.112	4.112	4.146	4.149	4.107	4.109	4.117
$p = 3$										
	10	4.186	4.387	4.177	4.183	4.246	4.444	4.238	4.441	4.348
3	20	3.952	3.996	3.938	3.940	3.969	4.014	3.955	4.002	3.971
	40	3.829	3.840	3.814	3.814	3.834	3.844	3.818	3.830	3.814
	10	4.472	4.564	4.452	4.460	4.472	4.564	4.451	4.551	4.505
6	20	4.308	4.329	4.282	4.285	4.308	4.329	4.281	4.305	4.293
	40	4.224	4.229	4.197	4.197	4.224	4.229	4.196	4.201	4.189
	10	4.649	4.702	4.623	4.629	4.598	4.651	4.567	4.627	4.649
10	20	4.530	4.543	4.499	4.501	4.516	4.528	4.482	4.496	4.499
	40	4.470	4.473	4.437	4.438	4.466	4.469	4.432	4.435	4.436
$p = 5$										
	10	4.941	5.370	4.938	4.941	5.022	5.444	5.019	5.444	5.324
3	20	4.589	4.681	4.581	4.584	4.613	4.704	4.605	4.699	4.657
	40	4.403	4.424	4.391	4.392	4.409	4.431	4.398	4.420	4.404
	10	5.116	5.286	5.107	5.113	5.116	5.286	5.107	5.282	5.237
6	20	4.880	4.918	4.862	4.865	4.880	4.918	4.861	4.903	4.885
	40	4.757	4.766	4.735	4.736	4.757	4.766	4.734	4.745	4.739
	10	5.236	5.324	5.221	5.227	5.170	5.260	5.151	5.248	5.278
10	20	5.068	5.088	5.045	5.047	5.049	5.070	5.023	5.046	5.055
	40	4.981	4.986	4.956	4.956	4.976	4.981	4.948	4.954	4.955

$$FAC(\widehat{\kappa}) \equiv FA(p,k,N,\widehat{t}_{1\cdot\chi};\alpha) = \frac{\widehat{t}_{1\cdot\chi} - t^*}{t^*},$$

$$MSAC(\widehat{\kappa}) \equiv MSA(p,k,N,\widehat{t}_{M\cdot\chi};\alpha) = \frac{\widehat{t}_{M\cdot\chi} - t^*}{t^*},$$

for $\alpha = 0.05$ and selected values (p,k,N), and Figures 3 and 4 present the relative errors of these approximations based on F approximation for $\alpha = 0.05$ and selected values (p,k,N), which are given by

$$FAF(\kappa) \equiv FA(p,k,N,t_{1\cdot F};\alpha) = \frac{t_{1\cdot F} - t^*}{t^*},$$

$$MSAF(\kappa) \equiv MSA(p,k,N,t_{M\cdot F};\alpha) = \frac{t_{M\cdot F} - t^*}{t^*},$$

Table 6. Continued.

Elliptical distribution(Contaminated normal; $\kappa = 1.78$)										
$p = 2$										
k	N	$t_{1\cdot\chi}$	$t_{1\cdot F}$	$t_{M\cdot\chi}$	$t_{M\cdot F}$	$\widehat{t}_{1\cdot\chi}$	$\widehat{t}_{1\cdot F}$	$\widehat{t}_{M\cdot\chi}$	$\widehat{t}_{M\cdot F}$	t^*
	10	3.475	3.604	3.440	3.455	3.772	3.891	3.757	3.882	3.673
3	20	3.427	3.455	3.398	3.402	3.551	3.579	3.534	3.563	3.471
	40	3.402	3.409	3.379	3.380	3.447	3.454	3.430	3.437	3.403
	10	4.081	4.143	4.056	4.063	4.081	4.143	4.051	4.120	4.053
6	20	3.955	3.969	3.925	3.927	3.955	3.969	3.921	3.938	3.919
	40	3.890	3.894	3.859	3.860	3.890	3.894	3.857	3.861	3.854
	10	4.509	4.544	4.498	4.500	4.258	4.296	4.219	4.263	4.364
10	20	4.310	4.318	4.289	4.290	4.206	4.215	4.165	4.175	4.231
	40	4.207	4.209	4.179	4.180	4.169	4.171	4.128	4.130	4.154
$p = 3$										
	10	3.856	4.073	3.828	3.846	4.236	4.435	4.228	4.431	4.189
3	20	3.780	3.827	3.755	3.760	3.945	3.990	3.931	3.979	3.863
	40	3.741	3.752	3.720	3.721	3.804	3.815	3.789	3.800	3.745
	10	4.472	4.564	4.455	4.462	4.472	4.564	4.451	4.551	4.474
6	20	4.308	4.329	4.285	4.287	4.308	4.329	4.281	4.305	4.279
	40	4.224	4.229	4.198	4.199	4.224	4.229	4.196	4.201	4.189
	10	4.910	4.960	4.905	4.906	4.607	4.659	4.576	4.636	4.759
10	20	4.666	4.678	4.652	4.653	4.536	4.548	4.501	4.515	4.585
	40	4.539	4.542	4.518	4.518	4.490	4.493	4.454	4.458	4.489
$p = 5$										
	10	4.494	4.961	4.477	4.492	5.017	5.440	5.015	5.440	5.162
3	20	4.353	4.450	4.334	4.341	4.594	4.686	4.586	4.680	4.529
	40	4.281	4.304	4.263	4.265	4.379	4.401	4.367	4.390	4.321
	10	5.116	5.286	5.109	5.114	5.116	5.286	5.107	5.282	5.180
6	20	4.880	4.918	4.865	4.868	4.880	4.918	4.861	4.903	4.856
	40	4.757	4.766	4.738	4.739	4.757	4.766	4.734	4.745	4.730
	10	5.570	5.653	5.568	5.569	5.174	5.263	5.155	5.252	5.396
10	20	5.243	5.262	5.235	5.236	5.064	5.084	5.039	5.062	5.144
	40	5.071	5.076	5.057	5.057	5.000	5.005	4.971	4.977	5.016

$$FAF(\widehat{\kappa}) \equiv FA(p,k,N,\widehat{t}_{1\cdot F};\alpha) = \frac{\widehat{t}_{1\cdot F} - t^*}{t^*},$$

$$MSAF(\widehat{\kappa}) \equiv MSA(p,k,N,\widehat{t}_{M\cdot F};\alpha) = \frac{\widehat{t}_{M\cdot F} - t^*}{t^*}.$$

It can be seen from Tables and Figures that the modified second approximation is close to the simulated value and the conservative approximation as the sample sizes are large. Also, it may be noted that the modified second approximation is still better in the case when the kurtosis parameter is estimated. Further, it can be seen from the simulation study that the simulated values for the case of $k = 6$ do not depend on the form of the distribution for the underlying populations. It may be also noted from Figures 1 and 2 that the approximate values are underestimated when k and N are small and κ is known. This phenomenon shows that asymptotic expansion procedure provides underestimated approximation for the small sample size. On the case of elliptical populations, it may be noted that approximate values by the modified second approximation Bonferroni procedure are closer to the simulated values than the first approximation for the moderate magnitude of sample size. In addition, we note that the magnitude of improvement is large for the case when k is large because the improvement by the modified second approximation procedure yields an underestimation for the first approximation which is more overestimated as k is large.

Next, to examine the effect of nonnormality on these approximations when the populations have not elliptical distributions, as an example, we consider the case when each population has the multivariate χ^2 distribution with five degrees of freedom where each component is an independent χ^2 random variable. In this case, there is no existence of the kurtosis parameter defined in Section 2 but the numerical value which corresponds to the estimation $\widehat{\kappa}$ can be obtained. Therefore, substituting this in the approximation formula, this result can be used as an approximation for the upper percentiles of $T^2_{\max}$ statistic under nonnormality. We consider the Monte Carlo simulation and approximate value in the same setup as that used in the above elliptical population case. Figure 5 shows the relative errors of the approximate values, $FAC(\widehat{\kappa})$, $MSAC(\widehat{\kappa})$, $FAF(\widehat{\kappa})$, and $MSAF(\widehat{\kappa})$. It may be expected from the Figure 5 that the modified second approximate value

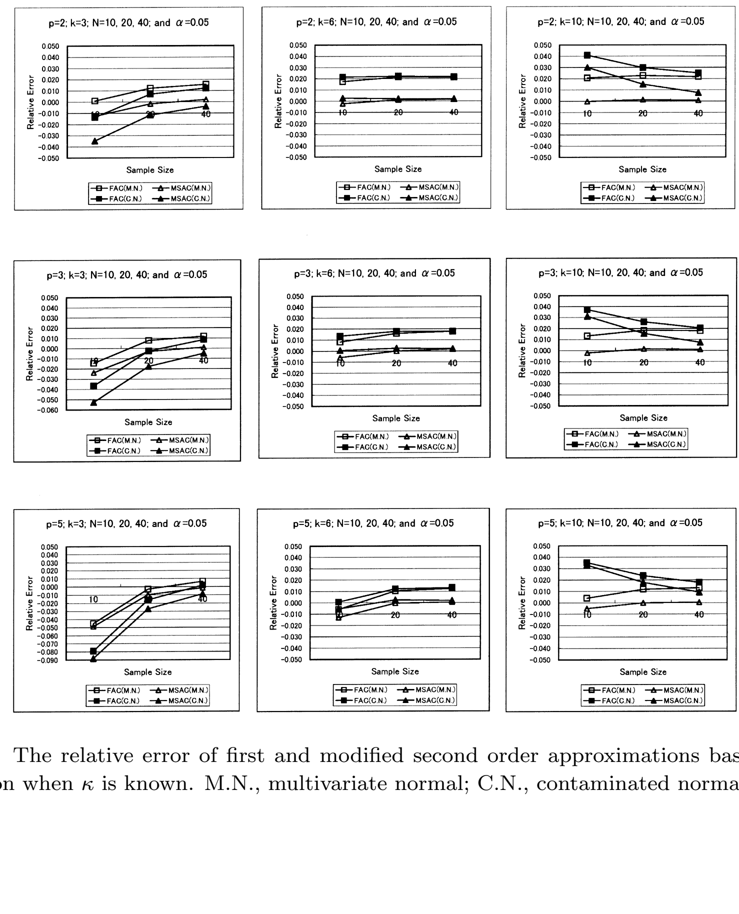

Figure 1. The relative error of first and modified second order approximations based on χ^2 distribution when κ is known. M.N., multivariate normal; C.N., contaminated normal($\varepsilon = 0.1$, $\sigma = 3$).

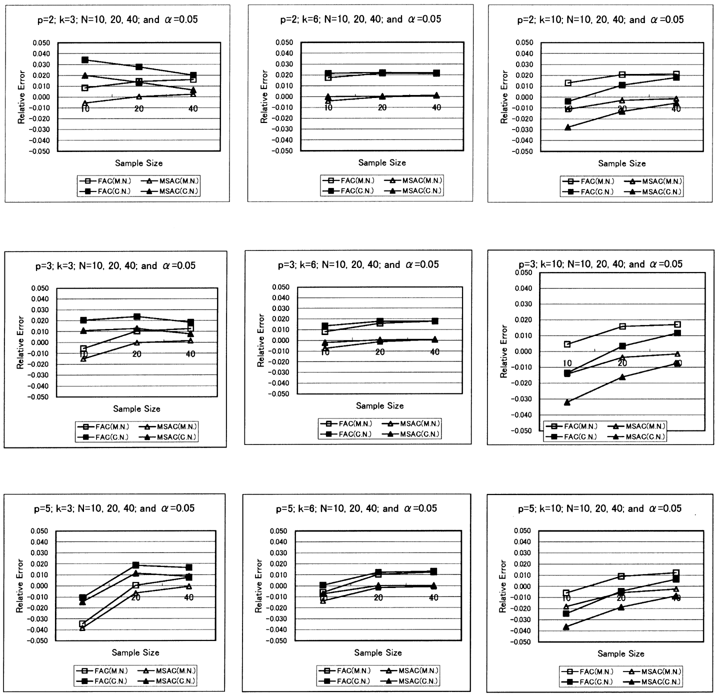

Figure 2. The relative error of first and modified second order approximations based on χ^2 distribution when κ is unknown.

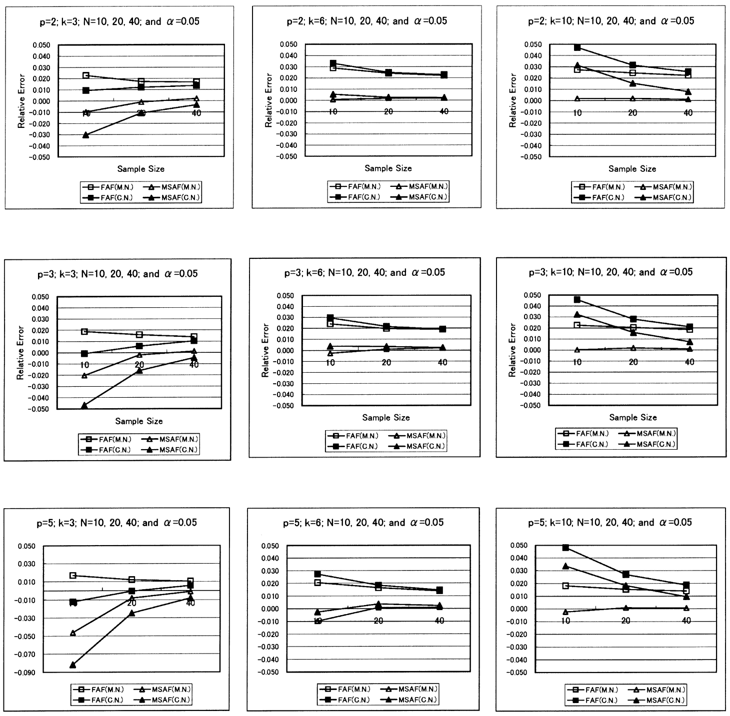

Figure 3. The relative error of first and modified second order approximations based on F distribution when κ is known.

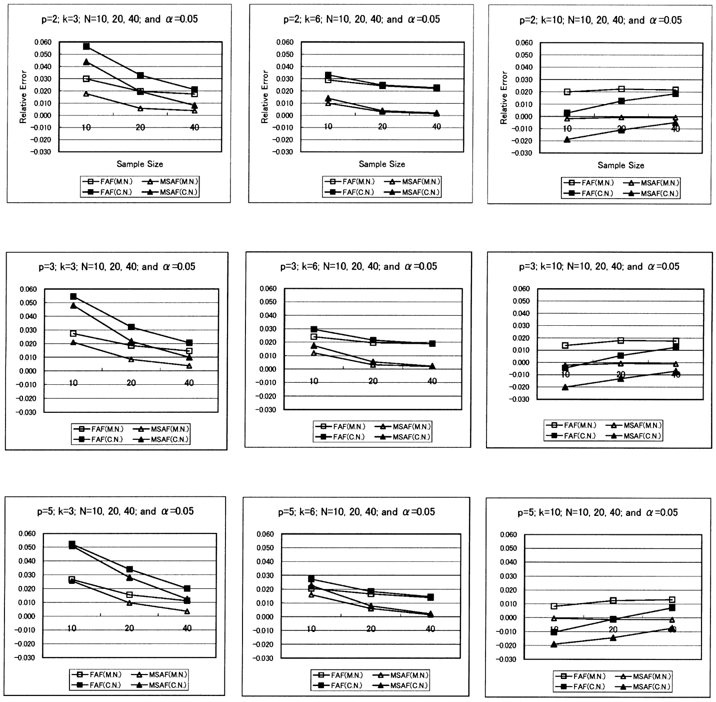

Figure 4. The relative error of first and modified second order approximations based on F distribution when κ is unknown.

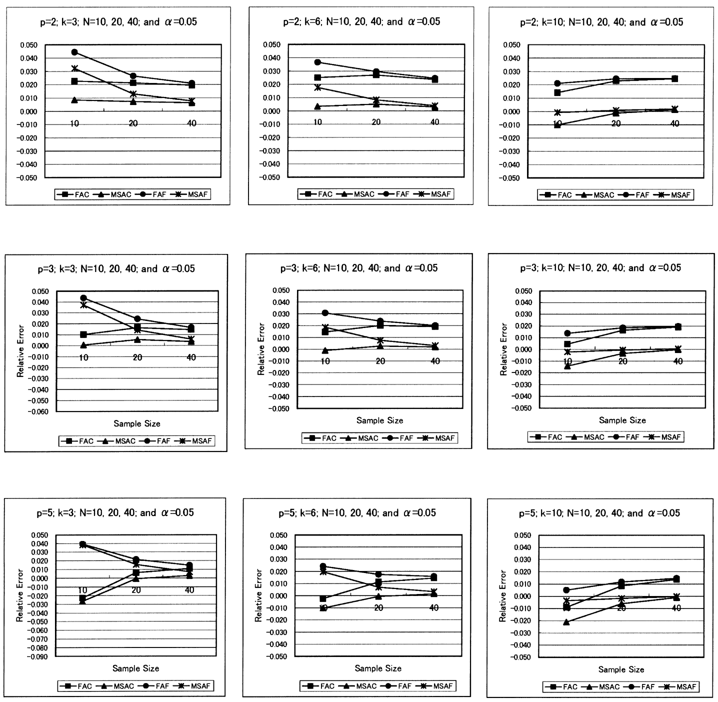

Figure 5. The relative error of first and modified second order approximations when each population has the multivariate χ^2 distribution$(d.f. = 5)$.

is useful for the simultaneous confidence intervals estimation for the pairwise multiple comparisons among mean vectors under the nonnormal distributions, though we need to examine how much these approximations are affected by departures from normality for other nonnormal distributions.

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