European Option Pricing under Fractional Brownian Motion with an Application to Realized Volatility

Takayuki Morimoto

Department of Mathematical Sciences, Kwansei Gakuin University, Japan
E-mail address: morimot@kwansei.ac.jp

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This study investigates European option pricing under fractional Brownian motion (fBm) and applies it to realized volatility (RV). The RV measure is selected because it uniquely exhibits simultaneous stationarity and long-range dependency properties in financial time series, as shown in our empirical study. Meanwhile, the Black-Scholes differential equation is not well defined when the underlying assets follow fBm with the Hurst exponent $H \neq \frac{1}{2}$ because fBm is not a semimartingale. Thus, we compute the European option prices using a previously proposed fractional Black-Scholes formula. Our empirical study is conducted on Tokyo Stock Price Index data from January 06, 1997 to December 30, 2013 with a sample size of 4177.

Key words: Fractional Brownian Motion, Realized Volatility, Long-range Dependency, Fractional Black-Scholes Formula, Tokyo Stock Price Index Data

1. Introduction

Fractional Brownian motion (fBm) was mathematically introduced in 1940 by Kolmogorov as a method to generate Gaussian spirals in a Hilbert space as stated in Bardet and Bertrand (2007). The applicability of fBm to real world has been empirically suggested by ancient sages.

In 1906, a young Englishman named Harold Edwin Hurst arrived in Cairo. It was to have been a short story. But it lasted sixty-two years and ended with his solving one of the great mysteries of the pharaohs—and, inadvertently, providing a clue to the way financial markets work.


Eventually, Hurst found a long-term relationship between the volatile rain on the Nile River and drought conditions. From these results, he determined the optimum dam sizing and published his seminal paper (Hurst, 1951). Subsequently, long-run dependences in financial time series have been frequently reported. Mandelbrot and Van Ness (1968) first investigated a self-similar process with a long-range dependent incremental process, known as fractional Brownian motion (fBm). Meanwhile Granger (1980), Granger and Joyeux (1980), and Hosking (1981) proposed an autoregressive fractionally integrated moving average (ARFIMA) model for a discrete time series with a long memory. For details of these concepts, see Beran (1994) and Baillie (1996).

Lo (1991) indicated that long-memory components in asset returns are crucial to many paradigms of modern financial economics. For example,

1) If stock returns are long-range dependent, the optimal consumption-savings and portfolio decisions may become extremely sensitive to the investment horizon.

2) The pricing of derivative securities (such as options and futures) by martingale methods is problematic, since the commonly employed class of continuous-time stochastic processes is inconsistent with long-term memory. Traditional tests of the capital asset pricing model and the arbitrage pricing theory are invalidated for persistent time series, because these series cannot be evaluated by typical statistical inferences.

3) The conclusions of more recent tests of “efficient” market hypotheses and stock market rationality also heavily rely on the presence or absence of long-term memory.

This study focuses on implication 2, which indicates that pricing options by martingale methods is inconsistent with long-term memory.

Therefore, we first study the definitions of long-memory processes in discrete time and the autoregressive integrated moving average (ARIMA) model with a fractional differencing parameter. Second, we introduce the $R/S$ statistic proposed by Hurst (1951) and apply it to long-range dependency testing. Third, we construct a realized volatility (RV) measure for our empirical study and unit root tests for testing stationarity. Finally, we investigate European option pricing subjected to fBm and apply it to RV. Our empirical study uses Tokyo Stock Price Index (TOPIX) data from January 06, 1997 to December 20, 2013. The sample size is 4177.

Since the seminal paper of Black and Scholes (1973), numerous papers have described option pricing by a partial differential equation. However, the Black-Scholes dif-
ferential equation is poorly defined when the underlying assets follow fBm with the Hurst exponent \( H \neq \frac{1}{2} \) because fBm is not a semimartingale, and the usual Itô integral cannot be integrated with respect to fBm (Biagini et al., 2008). Hu and Øksendal (2003) and Elliott and Van Der Hoek (2003) defined a new stochastic integral based on Wick products and Skorohod integration, and showed that it prevents arbitrage opportunities. However, although these models guarantee no arbitrage opportunities, they have no natural economic interpretation (Björk and Hult, 2005). For this reason, we compute the European option prices by the fractional Black-Scholes formula proposed by Norros et al. (2001).

The remainder of this study is as follows. Section 2 briefly explains long-range dependency in discrete time. Sections 3 and 4 introduce the concept of fBm and option pricing under fBm. Section 6 presents our empirical result and explains the RV measure and unit root tests for testing stationarity. Section 7 concludes the study.

2. Long-range Dependency

In this section, we briefly survey the stochastic nature of long-range dependence (LRD) in a time series, following Minotani (2001).

The autocorrelations in stochastic processes with LRD, or long memory, decay slowly to zero. A process is called a short memory process if its autocorrelation function \( \rho \) at lag \( k \) takes the form

\[
|\rho_k| \leq c a^k, \quad c, a \text{ are constants},
\]

and a long memory process if \( \rho_k \) takes the form

\[
\rho_k \approx c k^{-d-1}, \quad 0 < d < \frac{1}{2},
\]

where \( d \in \mathbb{R} \) (Brockwell and Davis, 1991). For instance, \( X_t \) is expected to follow the first-order autoregressive model AR(1)

\[
(1 - \phi_1 L)X_t = \epsilon_t, \quad |\phi_1| < 1, \quad \epsilon_t \sim \text{iid}(0, \sigma^2),
\]

where \( \phi_1 \in \mathbb{R} \) and \( L \) denotes the lag operator), the autocovariance function of process \( \gamma_k \) becomes

\[
\gamma_k = \frac{\phi_1 \sigma^2}{1 - \phi_1^2} \to 0, \quad k \to \infty,
\]

and the autocorrelation function is

\[
\rho_k = \phi_1^k \to 0, \quad k \to \infty.
\]

Hence, AR(1) is found to be a short memory process. As another example, if \( X_t \) is expected to follow the autoregressive moving average (ARMA) model of order \((p, q)\), denoted by ARMA\((p, q)\), and the process is presumed stationary and invertible, the autocovariance function becomes

\[
\gamma_k \approx c \lambda_1^k \to 0, \quad k \to \infty, \quad |\lambda_1| < 1,
\]

where \( \lambda_1, \ldots, \lambda_p \) are the roots of the characteristic equation of the AR\((p)\) and \( \lambda_1 \) is closest to 1. Hence ARMA\((p,q)\) is also a short memory process.

On the other hand, \( X_t \) has a long memory property if its autocovariance function is expressed as Lo (1991)

\[
\gamma_k \sim \begin{cases} k^{2H-2} L(k), & \frac{1}{2} < H < 1, \\ -k^{2H-2} L(k), & 0 < H < \frac{1}{2}, \end{cases}
\]

where \( H \) denotes the Hurst exponent and \( L(k) \) is an arbitrary function such as \( \lim_{t \to \infty} L(t) = 1 \).

2.1 ARFIMA model

The ARIMA process takes the form

\[
\phi(L)(1 - L)^d X_t = \theta(L)\epsilon_t,
\]

where \( \phi, \theta \in \mathbb{R} \), and \( d \) is the difference parameter, which must be integer valued. The ARFIMA model extends the ARIMA model to allow fractional order of differencing \( d \) (Granger, 1980; Granger and Joyeux, 1980; Hosking, 1981). An ARFIMA\((p, d, q)\) has the following \( d \)-dependent properties (Chen et al., 2008; Das and Pan, 2011).

1) If \( d = 1/2 \), the ARFIMA\((p, -1/2, q)\) process is stationary but non-invertible.
2) If \(-1/2 < d < 0\), the ARFIMA\((p, d, q)\) process has short memory and monotonically and hyperbolically decays to zero.
3) If \( d = 0 \), the ARFIMA\((p, 0, q)\) process can be white noise.
4) If \( 0 < d < 1/2 \), the ARFIMA\((p, d, q)\) process is a long-memory stationary process. This model appropriately describes LRD. The autocorrelation of a LRD time series slowly decays as a power law function.
5) If \( d = 1/2 \), the ARFIMA\((p, 1/2, q)\) process is a discrete time \( 1/f \) noise.

2.2 R/S statistic

Hurst (1951) and Mandelbrot (1972) developed the R/S statistic for testing whether LRD exists in a time series. First, we define the range \( R \) of \( X \) at time \( T \) as

\[
R_T = \max_{1 \leq k \leq T} \sum_{t=1}^{k} (X_t - \bar{X}) - \min_{1 \leq k \leq T} \sum_{t=1}^{k} (X_t - \bar{X}),
\]

where \( \bar{X} \) denotes the sample mean of \( X_t \). Second, we estimate the sample standard deviation of \( X_t \) as

\[
S_T = \left[ \frac{1}{T} \sum_{t=1}^{T} (X_t - \bar{X})^2 \right]^{1/2}.
\]

The quantity \( Q = R_T / S_T \) is called the R/S statistic. Hurst (1951), Mandelbrot (1972), and Lo (1991) showed that

\[
p \lim_{T \to \infty} T^{-H} (R_T / S_T) = \text{constant}.
\]

They also approximated the above equation as

\[
\log[E(R_T / S_T)] \approx \text{constant} + H \log(T).
\]

Rearranging this formula, the Hurst exponent \( H \) is approximated by

\[
H \approx \frac{\log(E(R_T / S_T))}{\log(T)}.
\]
The properties of the Hurst exponent \( H \) depend on its value. In the original theory, \( H = 1/2 \) implies an independent process, whereas \( 1/2 < H \leq 1 \) implies a persistent time series, which is characterized by long memory effects. Theoretically, what happens today forever influences the future. \( 0 \leq H < 1/2 \) signifies anti-persistence. An anti-persistent system covers less distance than a random one, because it self-reverses more frequently than a random process. In addition, the Hurst exponent \( H \) is directly related to the fractional differencing operator \( d \) of the ARFIMA model as

\[
d = H - \frac{1}{2}.
\]

In fact, when testing for LRD in \( X_t \), we require the sample distribution of the \( R/S \) statistic. Again, we introduce \( Q_T \) as

\[
Q_T = \frac{R_T}{S_T} = \frac{1}{S_T} \left[ \max_{1 \leq k \leq T} \sum_{t=1}^{k} (X_t - \bar{X}) - \min_{1 \leq k \leq T} \sum_{t=1}^{k} (X_t - \bar{X}) \right].
\]

Assuming that \( X_t \sim \text{iid} \) is true, we have

\[
\frac{1}{\sqrt{T}} Q = V,
\]

where \( \Rightarrow \) denotes convergence in law, \( V \) is the range of Brownian bridge in \([0, 1] \), \( \text{E}(V) = \sqrt{T} \) and \( \text{Var}(V) = \frac{1}{2} \pi (\pi - 3) \); see Lo (1991).

### 2.3 Lo’s modified \( R/S \) statistic

The type-I error probability is known to exceed the nominal size of the \( R/S \) statistic. Consequently, a short memory process may be incorrectly deemed as a long memory one. To correct this problem, Lo (1991) proposed the following modified \( R/S \) statistic:

\[
Q^* = \frac{R_T}{\hat{S}_T(q)},
\]

where \( \hat{S}_T(q) = S_T^2 + 2 \sum_{j=1}^{q} \omega_j(q)c_j, \ c_j = \hat{y}_j \) and \( \omega_j(q) = 1 - \frac{q}{\sigma^2} \) for \( q < T \). The \( Q^* \) and \( Q \) differ in one respect only; the divisor of \( Q^* \) is \( \hat{S}_T(q) \), whereas that of \( Q \) is \( S_T \). For instance, when \( X_t \) follows an AR(1) process, \( q = 1 \) and

\[
\delta_1^2(1) = S_T^2 + c_1.
\]

Moreover, since \( Q^* < Q \) by virtue of \( c_1 > 0 \), we have \( \delta_1^2(1) > S_T^2 \). \( Q^* \) also has the following asymptotic property:

\[
\frac{1}{\sqrt{T}} Q^* \Rightarrow V,
\]

where \( V \) is the range of the Brownian bridge in \([0, 1] \) as stated before. However, we notice that a large lag \( q \) may lower the power of the test, indicating that if \( q \) is oversized, \( Q^* \) cannot properly detect the LRD in a time series. Hence, the right choice of \( q \) is essential in Lo’s method (Teverovsky et al., 1999).

### 3. Fractional Brownian Motion

A stochastic process \( Z \) on the filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) is called fBm \( \{Z_t\}_{t \geq 0} \) with Hurst exponent \( H \in (0, 1) \) if the following conditions are satisfied:

1. \( Z_t \) has strictly stationary increments, that is, \( Z_{t+s} - Z_t, s \geq 0 \) is strictly stationary.
2. \( Z_0 = 0 \) and \( \text{E}[|Z_t|] = 0 \) for all \( t \) almost surely.
3. \( \text{E}[|Z_t|^2] = |t|^{2H} \) for all \( t \) and \( H \in (0, 1) \).
4. \( Z_t \) follows a Gaussian distribution.
5. \( Z_t \) is almost surely continuous.

The stochastic process is defined by

\[
Z_t - Z_s = c_H \left[ \int_s^t (t-u)^{-1/2} dw_u \right. + \left. \int_{-\infty}^s (t-u)^{-1/2} -(s-u)^{-1/2} dw_u \right],
\]

(2)

where \( w_t \) is a standard Brownian motion (Bm) and \( c_H \) is given by

\[
c_H = \left[ \frac{2H \Gamma(\frac{3}{2} - H)}{\Gamma(1 + \frac{3}{2}) \Gamma(2 - 2H)} \right]^{1/2},
\]

with \( \Gamma \) being the gamma function. Setting \( s = 0 \) in Eq. (2) we can write

\[
Z_t = c_H \left[ \int_0^t (t-u)^{-1/2} dw_u \right. + \left. \int_{-\infty}^0 (t-u)^{-1/2} -(-u)^{-1/2} dw_u \right].
\]

If \( H = \frac{1}{2} \), fBm reduces to standard Brownian motion.

\[
Z_t = c_{\frac{1}{2}} \left[ \int_0^t (t-u)^{-1/2} dw_u \right. + \left. \int_{-\infty}^0 (t-u)^{-1/2} -(-u)^{-1/2} dw_u \right],
\]

\[
= \int_0^t dw_u = w_t,
\]

where

\[
c_{\frac{1}{2}} = \left[ \frac{2 \cdot \frac{1}{2} \Gamma(\frac{1}{2} - \frac{1}{2})}{\Gamma(\frac{1}{2} + \frac{3}{2}) \Gamma(2 - 2 \cdot \frac{1}{2})} \right]^{1/2} = 1.
\]

As described in the previous section, fBm with \( 1/2 < H \leq 1 \) implies a persistent time series and \( 0 \leq H < 1/2 \) signifies antipersistence.

### 3.1 A fractional Brownian market model

We introduce a riskless asset \( A_t \) and a risky asset \( S_t \) by means of a geometric fBm driven by \( dA_t = rA_t dt \) and \( dS_t = \mu S_t dt + \sigma S_t dB_t^H \), respectively. The parameters for the riskless interest rate \( r \) as well as the drift \( \mu \) and the volatility \( \sigma \) are constant. The stochastic differential equation can be interpreted in multiple ways depending on the chosen stochastic integration calculus, which are pathwise integration, and in contrast, Wick-based integration (see Rostek and Schöbel, 2013).
Numerous articles have been published choosing fractional Brownian motion as an underlying diffusive process. There has been an ongoing discussion about the usage of fBm within financial models since Rogers (1997). Some publications also discuss market microstructure foundations of fBm while most of the literature focuses on arbitrage and its exclusion. Shiryaev (1998) constructed an explicit arbitrage strategy within the fractional market setting based on pathwise integrals. Duncan et al. (2000) provided a stochastic integration calculus with respect to fBm based on the Wick product. For more detailed information, see Rostek and Schöbel (2013).

### 3.2 Wick-Itô integral

Let $w$ be a Brownian motion and $F$ be a left-continuous, adapted and locally bounded deterministic process. The mean squares of the Riemann-Stieltjes sums converge to a random variable called the Itô integral of $F$ with respect to $w$, which is defined up to time $T$:

$$\int_0^T F(s)dw_s = \lim_{n \to \infty} \sum_{i=1}^{n} F(t_i)(w_{t_{i+1}} - w_{t_i}).$$

However, if the integrand is no longer deterministic, convergence in the mean square sense is not necessarily given, see Rostek (2009). Therefore, following Duncan et al. (2000), we replace the ordinary multiplication within the Riemann-Stieltjes sums with a different multiplicative concept called the Wick product (denoted by a diamond symbol $\diamond$). The fractional Wick-type integral, called the Wick-Itô integral, is the limit of the according sequence of Riemann-Stieltjes sums:

$$\int_0^T F(s)dw_s^H = \lim_{n \to \infty} \sum_{i=1}^{n} F(t_i) \diamond (w_{t_{i+1}}^H - w_{t_i}^H).$$

The Wick-Itô integral cannot exhibit martingale behavior since the integrator of fBm is not a semimartingale. Meanwhile, the Wick-Itô approach is formally compatible with classical Brownian theory when pricing options (Rostek, 2009).

Following Biagini et al. (2008), we now explain the Wick product in the above equation. First, we define the $n$th-order Hermite polynomial:

$$h_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}), \quad n \geq 0,$$

and the $n$th order Hermite function:

$$\tilde{h}_n := \pi^{-1/4}(n!)^{-1/2}2^{-n/2}h_n(x)e^{-x^2/2}, \quad n \geq 0,$$

also we have

$$H_n(\omega) := \Pi_{i=1}^n h_{\alpha_i}(<\tilde{h}_i, \omega>),$$

where $\omega \in \Omega$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, and $\alpha_i \in \mathbb{N} \cup \{0\}$. Let two random variables $F(\omega), G(\omega) \in L^2(\Omega)$ be chaotically expanded as

$$F(\omega) = \sum_{a} c_a H_a(\omega) \quad \text{and} \quad G(\omega) = \sum_{\beta} d_\beta H_\beta(\omega),$$

respectively, where $c_a, d_\beta \in \mathbb{R}$, $\beta = (\beta_1, \ldots, \beta_n)$ and $\beta_i \in \mathbb{N} \cup \{0\}$. The Wick product is then defined as

$$(F \diamond G)(\omega) = \sum_{a, \beta} c_a d_\beta H_{a+\beta}(\omega).$$

The Radon-Nikodym derivative for fBm is provided in Appendix A and for the discrete approximation of fBm, see Appendix B.

Hu and Øksendal (2003) and Elliott and Van Der Hoek (2003) implemented the Wick product into the definitions of the portfolio value and/or self-financing property. The seemingly encouraging result of a fractional Black-Scholes market excluding arbitrage provided by Hu and Øksendal (2003) and Elliott and Van Der Hoek (2003) entailed further models. Although this pricing approach grew in popularity, some serious concerns questioned the usage of Wick products beyond pure integration theory. The suitability of Wick-based definitions of fundamental economic concepts was first doubted by Sottinen and Esko (2003).

#### 3.3 Option pricing with fBm

When the underlying assets follow fBm with $H \neq \frac{1}{2}$, fBm is not a semimartingale, and the usual Itô integral cannot be integrated with respect to fBm (Biagini et al., 2008). Under this circumstance, the Black-Scholes differential equation is ill-defined. To circumvent this problem, Hu and Øksendal (2003) and Elliott and Van Der Hoek (2003) defined a new stochastic integral based on Wick products and Skorohod integration, which ensures the absence of arbitrage opportunities. However, as already mentioned, these models lack a natural economic interpretation (Björk and Hult, 2005). We introduce self- and Wick-financing portfolios following Björk and Hult (2005) in Appendix C.

#### 3.4 Fractional Black-Scholes formula

In the study, we compute European option prices using the fractional Black-Scholes formula proposed by Norros et al. (1999).

Suppose that a stock price follows geometric Brownian motion. Then, the price of a European call option $C_{bs}$ is given by the celebrated Black-Scholes formula

$$C_{bs} = s \Phi(d_1) - k \exp(-rT) \Phi(d_2),$$

where $s$ is the value of the underlying asset, $k$ is the strike price of the option, $t$ is the expiry time of the option, $r$ is the risk-free interest rate, and $\sigma$ is the volatility of the underlying asset. $\Phi(\cdot)$ represents the cumulative distribution function of a standard normal variable, and $d_1$ and $d_2$ are given by $d_1 = \frac{\ln(s/k) + (r + \sigma^2/2)T}{\sigma \sqrt{T}}$ and $d_2 = d_1 - \sigma \sqrt{T}$. As shown by Norros et al. (1999), one can define a centered Gaussian martingale (the fundamental martingale) that generates the same filtration as fBm (Kozlowski, 2012). Since the filtration (rather than the stochastic process itself) represents the information provided by the market, this martingale may reasonably be used for option pricing. By this approach, formulas analogous to the classical Black-Scholes formulas are easily obtained, and coincidence is achieved by setting the Hurst index to 1/2. The fractional European call option price $C_{fbs}$ is given by

$$C_{fbs} = s \Phi(d_1) - ke^{-rT} \Phi(d_2),$$

where $d_1 = \frac{\ln(s/k) + (r + \sigma^2/2)T}{\sigma \sqrt{T}}$ and $d_2 = d_1 - \sigma \sqrt{T}$. If $H = \frac{1}{2}$, we find that $C_{bs} = C_{fbs}$. That is, the differences between the fractional and the standard Black-Scholes prices are solely determined by the value of the
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Fig. 1. Simulated fBm with $H = 0.50$.

Fig. 2. Simulated fBm with $H = 0.75$.

long memory parameter $H$ (i.e., the Hurst exponent). The fractional European put option price $P_{\text{fbs}}$ is also determined by the standard put-call parity relation

$$P_{\text{fbs}} = C_{\text{fbs}} + k e^{-rt} - s.$$

4. Simulation Study

In this section, we examine how the fractional European call option prices are evaluated in simulation studies. Figures 1 and 2 are examples of fBm paths with $H = 0.50$ and $H = 0.75$, respectively. Simulations were performed using the dvfBm package function circFBSM() in the R language. As noted above, $H = 0.50$ (Fig. 1) describes standard Brownian motion, while $H = 0.75$ (Fig. 2) describes a process with a long memory property. Generation of the fractional Brownian paths is detailed in Coeurjolly (2000).

Figures 3 and 4 show the European call option prices calculated by the fractional Black-Scholes formula introduced above. The Hurst parameters are set to $H = 0.50$ and $H = 0.75$ respectively, the current stock price $s = 100$, the volatility $\sigma = 0.5$, and the risk-free interest rate $r = 0.1$.

Since Figs. 3 and 4 are visually very similar, we clarify their differences in Figs. 5 and 6. Figure 5 shows how the European call option prices differ between $H = 0.50$ and $H = 0.75$ with other parameters fixed at $s = 100$, $\sigma = 0.5$, and $r = 0.1$. According to this figure, the price difference is enhanced around the at-the-money (here denoted by the strike price $k = 100$), and widens as the time to maturity reduces. Figure 6 shows how the European call option prices as the Hurst exponent $H$ varies from 0 to 1. Other parameters are fixed at $s = 100$, $\sigma = 0.5$, $r = 0.1$, and $k = 100$. From this figure, we observe that the difference greatly increases as the time to maturity reduces and the Hurst exponent $H$ increases.

5. Empirical Study

In this section, we examine the long memory property of real financial data such as time series of stock prices, log-
return, and volatility. We employ the daily closing price data of the Tokyo stock price index (TOPIX) from January 1997 to December 2013, denoting its value at time \( t \) by \( S_t \). The daily logarithmic return at time \( t \) is defined as

\[
    r_t = \log S_t - \log S_{t-1}.
\]

Our analysis also includes the RV defined at time \( t \) as the sum of the intraday squared returns (Andersen et al., 2001):

\[
    RV_t = \sum_{i=1}^{n_t} r_{i,t}^2,
\]

where \( r_{i,t}^2 \) denotes a squared log-return (the \( i \)th observation on day \( t \)) and \( n_t \) is the number of data points in \( t \). Regarding the underlying log-price process as the continuous martingale part in a semimartingale model setup, the RV can be viewed as a proxy variable of the integrated variance calculated from the intraday full high-frequency log-returns. Consequently, the RV estimation requires the full high-frequency data over 24 h as a daily volatility measure. However, the Japanese stock market is divided into two sessions by a lunch break, i.e., the morning session lasts from 09:00 to 11:00 and the afternoon session from 12:30 to 15:00. Thus, we adopt the weighted RV proposed by Masuda and Morimoto (2012), which is a modified version adjusted to the Japanese market (Hansen and Lunde, 2005). The weighted RV with estimated optimal weights \( \lambda_1, \lambda_2, \lambda_3 \) and \( \lambda_4 \) is defined by

\[
    wRV_t = \lambda_1 y_{1,t}^2 + \lambda_2 y_{2,t}^2 + \lambda_3 y_{3,t}^2 + \lambda_4 y_{4,t}^2,
\]

where \( y_{1,t}^2, y_{2,t}^2, y_{3,t}^2, \) and \( y_{4,t}^2 \) denote the square of the close-to-open return, the RV in the morning session, the square of the lunch break return, and the RV in the afternoon session, respectively, on the \( t \)th day. Hereafter, we replace the weighted RV \( wRV \) by \( RV \) for notational simplicity. In addition, we set the sampling frequency to 1 min, the minimum observation interval of the Japanese stock market. The resulting sample sizes of the morning and afternoon sessions are 120 and 150, respectively.

### 5.1 Data description

In the empirical analysis, we first describe the three time series data \( S_t, r_t \) and \( RV_t \) discussed above. Figure 7 depicts the paths of \( S_t, r_t \) and \( RV_t \) over the sample period, and Table 1 presents the descriptive statistics of these data. The null hypothesis is that the data are independently distributed. According to the Ljung-Box (10) statistics for serial correlation in Table 1, we cannot reject the null hypothesis for

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**Table 1. Summary statistics of TOPIX data (1997–2013).**

<table>
<thead>
<tr>
<th></th>
<th>( S_t )</th>
<th>( r_t )</th>
<th>( RV_t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>1173.374</td>
<td>−0.0000</td>
<td>0.01025</td>
</tr>
<tr>
<td>Median</td>
<td>1145.760</td>
<td>0.0002</td>
<td>0.00716</td>
</tr>
<tr>
<td>Maximum</td>
<td>1816.970</td>
<td>0.1286</td>
<td>0.39879</td>
</tr>
<tr>
<td>Minimum</td>
<td>695.510</td>
<td>−0.1001</td>
<td>0.00001</td>
</tr>
<tr>
<td>Std. Dev.</td>
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<td>0.0141</td>
<td>0.01489</td>
</tr>
<tr>
<td>Skewness</td>
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<td>−0.2925</td>
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<tr>
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<td>8.402</td>
<td>223.907</td>
</tr>
<tr>
<td>Obs.</td>
<td>4177</td>
<td>4177</td>
<td>4177</td>
</tr>
<tr>
<td>LB(10)</td>
<td>41133.82×</td>
<td>22.47</td>
<td>8548.13×</td>
</tr>
</tbody>
</table>

Note that LB(10) denotes the Ljung-Box test statistics at lag 10 and × indicates the rejection of the null hypothesis that the process is not autocorrelated.
Fig. 8. Sample autocorrelation functions of $S_t$, $r_t$ and $RV_t$ time series in the TOPIX data (1997–2013).

Table 2. Results of $t$-ratio $\hat{r}_m$ estimated by the DF test.

<table>
<thead>
<tr>
<th>Assumed models</th>
<th>$S_t$</th>
<th>$r_t$</th>
<th>$RV_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AR(1)</td>
<td>-0.632</td>
<td>-63.28</td>
<td>-24.68</td>
</tr>
<tr>
<td></td>
<td>(0.421)</td>
<td>(0.001)</td>
<td>(0.001)</td>
</tr>
<tr>
<td>AR(1) with drift</td>
<td>-1.929</td>
<td>-63.27</td>
<td>-31.05</td>
</tr>
<tr>
<td></td>
<td>(0.328)</td>
<td>(0.001)</td>
<td>(0.001)</td>
</tr>
<tr>
<td>TS</td>
<td>-1.796</td>
<td>-63.28</td>
<td>-31.85</td>
</tr>
<tr>
<td></td>
<td>(0.694)</td>
<td>(0.001)</td>
<td>(0.001)</td>
</tr>
</tbody>
</table>

Note that TS represents trend-stationary and $p$-values for the null hypothesis are reported in parentheses.

Table 3. Results of the $R/S$ analysis.

<table>
<thead>
<tr>
<th>Used methods</th>
<th>$S_t$</th>
<th>$r_t$</th>
<th>$RV_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hurst-Mandelbrot</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\hat{\nu}$</td>
<td>1245.19</td>
<td>66.35</td>
<td>561.00</td>
</tr>
<tr>
<td>$\hat{H}$</td>
<td>0.855</td>
<td>0.503</td>
<td>0.759</td>
</tr>
<tr>
<td>Lo</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\hat{\nu}$</td>
<td>880.86</td>
<td>65.68</td>
<td>440.13</td>
</tr>
<tr>
<td>$\hat{H}$</td>
<td>0.813</td>
<td>0.502</td>
<td>0.730</td>
</tr>
</tbody>
</table>

$r_t$ at the 0.01 significance level, but the null hypothesis for $S_t$ and $RV_t$ is rejected at this level. Thus, the $S_t$ and $RV_t$ series show apparent serial correlations. As a graphical verification, we present sample autocorrelation functions for $S_t$, $r_t$, and $RV_t$ in Fig. 8. The correlogram impressively shows that the sample autocorrelation functions of both $S_t$ and $RV_t$ slowly decay, whereas that of $S_t$ is one at all lags.

5.2 Stationarity and long-range dependency

In this subsection, we examine the stationarity and long-range dependency of the series before setting the pricing options under fBm.

First, we perform a unit root test on the series, namely the Dickey-Fuller (DF) test proposed by Dickey and Fuller (1979). The unit root problem in a time series arises when either the autoregressive or moving average polynomial of an ARMA model has a root on or near the unit circle (Brockwell and Davis, 2002). We provide a brief theoretical explanation of the unit root test in Appendix D since a unit root in either of these polynomials has important implications for modeling. Table 2 shows the $t$-ratio $\hat{r}_m$ estimated by the DF test for each time series analyzed by the two models. While the $S_t$ exhibits no obvious stationarity property, the $r_t$ and $RV_t$ series are probably stationary processes.

Second, to examine the LRD of the data, we conducted an $R/S$ analysis using the Hurst-Mandelbrot and the Lo methods introduced in the previous section. The results of this analysis are presented in Table 3. The Hurst exponent $H$ of the $r_t$ by each method is approximately 0.5, implying that the process follows a standard Brownian motion. In contrast, the $S_t$ and $RV_t$ series are likely to have long-range dependency since their Hurst exponents lie within $[0.5, 1]$. In addition, we estimated the memory parameter $d$ in the ARIMA model, adopting the Sperio estimator proposed by Reisen (1994). Table 4 shows the estimated $d$ and $H$ for each time series, calculated by Equation (1). All the estimated Hurst exponents in Table 4 are slightly higher than those in Table 3.

5.3 Option pricing under fBm

Finally, we examined option pricing under fBm by the method of Norros et al. (1999). We confined this analysis to the $RV_t$ time series since the $RV_t$ data exhibit simultaneous stationarity and long-range dependency properties, as shown in Table 5.

Figure 9 shows how the European call option prices differ between $H = 0.50$ and $H = 0.7592$, estimated by the Hurst-Mandelbrot method. For comparison, the price differences between $H = 0.50$ and $H = 0.7301$ estimated by Lo’s method are presented in Fig. 10. In both figures, the differences are enhanced around the at-the-money (here denoting the strike price $k = 100$), and increase as the time to maturity decreases. These figures are plotted identically to Fig. 5 in the simulation study, but they exhibit a distinctly different shape. These shape differences might be explained by the different values of the volatility parameters in the simulation and the empirical study ($\sigma = 0.5$ and $\sigma = 0.01489$, respectively). The volatility is well known as the most sensitive input parameter in pricing options.

<table>
<thead>
<tr>
<th>Table 4. Estimated results of the ARFIMA model.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimates $S_t$ $r_t$ $RV_t$</td>
</tr>
<tr>
<td>$d$</td>
</tr>
<tr>
<td>$\hat{H}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 5. Summary of the stationarity and long-range dependency result.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stationarity $S_t$ $r_t$ $RV_t$</td>
</tr>
<tr>
<td>Long-range dependency</td>
</tr>
</tbody>
</table>
Fig. 9. Differences in call prices between fixed $H = 0.50$ and $H = 0.7592$.

Fig. 10. Differences in call prices between fixed $H = 0.50$ and $H = 0.7301$.

5.4 Discussion

The classical $R/S$ statistic intrinsically depends on ranges chosen arbitrarily by researchers. According to Teverovsky et al. (1999), the Lo’s test, which is based on the modified $R/S$ statistic $V$, tends to be very conservative in rejecting the null hypothesis of non-long-range dependence. The test performs very well and gives the correct results when a series exhibits only short-range dependence. However, the test still tends to incorrectly accept the null hypothesis when the series is long-range dependent.

Hence we introduce alternative semi-parametric methods of estimating the Hurst exponent $H = d + \frac{1}{2}$ which are known as a class of local Whittle estimators mainly analyzed by Robinson (1995), Shimotsu and Phillips (2005) and Shimotsu (2010). We will now give a brief explanation of these methods following Beran et al. (2013) and Kumar (2014).

The first alternative is the local Whittle (LW) method proposed by Künsch (1987) and Robinson (1995), which assumes the behavior of the spectral density $f(\lambda)$, when $\lambda = 0$. Suppose that $X_t$ is a stationary process with spectral density

$$f_X(\lambda) \sim c_f |\lambda|^{1-2H} \text{ as } \lambda \to 0$$

Then, given $d \in \Theta \subseteq (-\frac{1}{2}, \frac{1}{2})$, the LW estimator of $d$ is defined by

$$\hat{d}_{LW} = \arg \min_{d \in \Theta} K_m(d),$$

where

$$K_m(d) = \log G_m(d) - d \left( \frac{2}{m} \sum_{j=1}^{m} \log \lambda_j \right),$$

and for the spectral density satisfying the equation, the Whittle log-likelihood function is given as:

$$\hat{c}_f = G_m(d) = \frac{1}{m} \sum_{j=1}^{m} I_{n,X}(\lambda_j) \lambda_j^{-2d},$$

Note that $\hat{H}$ is naively estimated from $\hat{H} = \hat{d} + 0.5$.

The empirical analog to the spectral density is the periodogram

$$I_{n,X}(\lambda) = \frac{1}{2\pi n} \left| \sum_{i=1}^{n} X_i e^{-ir\lambda} \right|^2 \text{ for } \lambda_j = \frac{2\pi j}{n}.$$
large while the values of $r$, and $RV$, are reasonable in comparison to the previous results. It has been considered that the relational expression $\tilde{H} = \tilde{d} + 0.5$ cannot be simply applied to a non-stationary time series such as $S_t$.

### 6. Conclusion

The objective of this paper was to study European option pricing under fBm with application to RV. We confined our study to the RV measure because the RV uniquely expressed both stationarity and long-range dependency properties in the financial time series, as shown in our empirical study. The Black-Scholes differential equation is not well defined when the underlying assets follow fBm with $H \neq \frac{1}{2}$ because the fBm is not a semimartingale. Thus, we computed European option prices using the fractional Black-Scholes formula proposed by Norros et al. (1999).

From our simulation study, we concluded that the European call option prices exhibited substantially different behaviors in the simulation and empirical studies. We attributed these differences to the different volatility parameters in the simulation and empirical studies. We attributed these differences to the different volatility parameters $\sigma = 0.5$ in the simulation and $\sigma = 0.01489$ in the empirical study. The volatility is well recognized as the most sensitive input parameter in pricing options.

### Acknowledgments

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### Appendix A. Radon-Nikodym Derivative for fBm

Let $X_t$ be a stochastic process driven by

$$X_t = Z_t + at,$$

where $a \in \mathbb{R}$ and $Z_t$ is an fBm under the probability measure $P$. We can change the measure of process $X_t$ via the Radon-Nikodym derivative of $Q$ with respect to $P$:

$$\frac{dQ}{dP} = \exp \left( -a M_t - \frac{1}{2} a^2 \langle M, M \rangle_t \right), \quad (A.1)$$

which yields a stochastic process $X$ without drift under the new probability measure $Q$. $M_t$ in Eq. (A.1) is given by

$$M_t = \int_0^t c_1 u^{1-H} (t-u)^{1-H} dw_u,$$

where $c_1 = \left[ 2H^+ \left( \frac{1}{2} - H \right) \Gamma H + \frac{1}{2} \right]^{-1}$ and $M_t$ is a martingale with independent increment and zero mean. The variance of $M_t$ is given by

$$\text{E}[M^2_t] = c_2^2 t^{2-2H}, \quad c_2 = \frac{c_H}{2H \sqrt{2 - 2H}}.$$

In terms of this expression, we can rewrite the Radon-Nikodym derivative (A.1) as

$$\frac{dQ}{dP} = \exp \left( -a M_t - \frac{1}{2} a^2 c_2^2 t^{2-2H} \right). \quad (A.2)$$

When $H = \frac{1}{2}$, Eq. (A.2) reduces to the change of measure formula in standard Brownian motion.

### Appendix B. Discrete Approximation of fBm

Discrete approximation problems for fBm can be categorized into three types depending on their $H$ values (Neuenkirch et al., 2010) as follows:

1. When $H > 1/2$, the Euler approximation converges at the rate $n^{-2H+1/2}$ for arbitrarily small $\epsilon > 0$ (Neuenkirch and Nourdin, 2007; Davie, 2008; Mishura and Shevchenko, 2008).
2. When $H = 1/2$, the approximation reduces to a discrete approximation of standard Brownian motion (Kloeden and Platen, 2010).
3. When $1/3 < H < 1/2$, the solution should be approximated by a Milstein-type scheme or a more sophisticated scheme (Lyons and Qian, 2003; Gubinelli, 2004). Moreover, it is easily seen that when $H < 1/2$, the standard Euler scheme does not converge as the step size reduces to zero, even in the one-dimensional case. In fact, consider the one-dimensional standard differential equation

$$dX_t = X_t dw_H^H, \quad X_0 = 1,$$

whose exact solution is given by $X_t = \exp(\int_0^t w_H^H)$. The Euler approximation of this equation at $t = 1$ is given by

$$X^{(n)}_1 = \prod_{k=0}^{n-1} \left( 1 + \left( w_{(k+1)/n}^H - w_{k/n}^H \right) \right).$$

For $n \in \mathbb{N}$ sufficiently large, a Taylor’s expansion gives

$$X_1 - X^{(n)}_1 = \exp(w_1^H) - \exp \left( \sum_{k=0}^{n-1} \log(1 + (w_{(k+1)/n}^H - w_{k/n}^H)) \right),$$

$$\exp(w_1^H) - \sum_{k=0}^{n-1} \frac{n-1}{2} \left| w_{(k+1)/n}^H - w_{k/n}^H \right|^2 + \rho_n,$$

where $\rho_n \xrightarrow{n \to \infty} 0$ for $n \to \infty$ for $H > 1/3$. Now, it is well known that

$$\sum_{k=0}^{n-1} \left| w_{(k+1)/n}^H - w_{k/n}^H \right|^2 \xrightarrow{n \to \infty} \infty,$$

when $H < 1/2$, so we have $X^{(n)}_1 \xrightarrow{n \to \infty} 0$. However, such one-dimensional equations converge under a Milstein-type scheme (Gradinaru and Nourdin, 2009).

Furthermore, Neuenkirch et al. (2010) demonstrated three different regimes for the convergence rate of the exact root mean square in the Euler scheme depending on the Hurst parameter $H \in (1/4, 1)$.

1. When $H < 3/4$, the exact convergence rate is $n^{-2H+1/2}$.
2. When $H = 3/4$, the exact convergence rate is $n^{-1/\log(n)}$.
3. When $H > 3/4$, the exact convergence rate is $n^{-1}$. 

---

**References**

Neuenkirch, A., and Nourdin, I. (2007). Discrete approximation problems for fBm can be categorized into three types depending on their $H$ values.


where \( n \) denotes the number of discretization subintervals.

Based on the above discussion, we now derive theorem 1 in Neuenkirch et al. (2010). Let \( B = (B^{(1)}, B^{(2)}) \) be a two-dimensional fBm with Hurst parameter \( H \in (1/4, 1) \) indexed by \( \mathbb{R} \). We approximate

\[
X_t = \int_0^T B_t^{(1)} dB_t^{(2)},
\]

(B.1)

by the Euler and a trapezoidal scheme based on equidistant discretizations. The standard Euler approximation of (B.1) is explicitly given by

\[
X^n_T = \sum_{i=0}^{n-1} B^{(1)}_{T/n} \left( B^{(2)}_{(i+1)T/n} - B^{(2)}_{iT/n} \right).
\]

(B.2)

From this expression, we can determine the exact \( L^2 \)-convergence rate of the Euler scheme.

**Theorem B.1 (Theorem 1 in Neuenkirch et al. (2010))**

Let \( X_T \) and its Euler approximation \( X^n_T \) be given by expressions (B.1) and (B.2), respectively. In addition, let

\[
\alpha_1(H) = c_0 + 2 \sum_{k=1}^{\infty} c_k H^{-k} \quad \text{and} \quad \alpha_2(H) = \frac{H^2(2H - 1)}{4(4H - 3)},
\]

where \( c_0 \) and \( c_k \) are constants defined in Neuenkirch et al. (2010). Then, \( \sum_{k=0}^{\infty} c_k \) is a convergent series if \( H \in (1/4, 3/4) \) and

\[
\mathbb{E}[|X_T - X^n_T|^2] \leq \begin{cases} 
\alpha_1(H) \cdot T^{4H} \cdot n^{-H+1} + o(n^{-H+1}) & \text{for } H \in (1/4, 3/4), \\
\frac{9}{16} \cdot T^3 \cdot \log(n)n^{-2} + o(\log(n)n^{-2}) & \text{for } H = 3/4, \\
\alpha_2(H) \cdot T^{4H} \cdot n^{-2} + o(n^{-2}) & \text{for } H \in (3/4, 1).
\end{cases}
\]

At the end of this section, we introduce a limit theorem for the asymptotic error distribution of the Euler scheme proposed by Neuenkirch et al. (2010).

**Theorem B.2 (Theorem 3 in Neuenkirch et al. (2010))**

Define \( X_T, X^n_T \) and \( \alpha_1(H), \alpha_2(H) \) as above. Moreover, let \( Z \) be a standard normal random variable. Then

1) Case \( 1/4 < H \leq 3/4 \). The following central limit theorems hold:

\[
\lim_{n \to \infty} n^{2H-1/2}(X_T - X^n_T) \xrightarrow{\mathcal{L}} \sqrt{\alpha_1(H)} T^{2H} \cdot Z,
\]

for \( H \in (1/4, 3/4) \) and

\[
\lim_{n \to \infty} n(\log(n))^{-1/2}(X_T - X^n_T) \xrightarrow{\mathcal{D}} \frac{3}{4\sqrt{8}} T^{3/2} \cdot Z,
\]

for \( H = 3/4 \) where \( \xrightarrow{\mathcal{L}} \) denotes convergence in law.

2) Case \( H > 3/4 \). Let \( R_1 \) and \( R_2 \) be two independent Rosenblatt processes (Neuenkirch et al., 2010 for the definition). Then we have

\[
\lim_{n \to \infty} n(X_T - X^n_T) \xrightarrow{\mathcal{D}} \sqrt{2\alpha_2(H)} T^{2H} \cdot (R_1 - R_2).
\]

**Appendix C. Portfolio Strategies**

**Self-financing portfolio** Consider a financial market with \( n+1 \) asset price processes \( S^0, S^1, \ldots, S^n \), and denote the corresponding vector process by \( S \). We consider an adapted portfolio process \( h = (h^0, h^1, \ldots, h^n) \) and define the value process \( V^h \) associated with \( h \) by the standard formula

\[
V^h_t = \sum_{i=0}^{n} h^i S^i_t = h_t S_t,
\]

where the equality between random variables is interpreted as equality \( P \)-almost surely. In continuous time, the self-financing concept becomes more complicated. However, we can specify a putative minimum requirement: that a buy-and-hold portfolio, i.e., a portfolio that remains constant over a fixed time interval, be self-financing over that interval. Let us consider the time interval \([t_0, t_1]\) and a portfolio \( h \) that is constant over that interval. At any time \( t \in [t_0, t_1] \) the portfolio value will be \( V_t = h_t S_t \). Since \( h_t \) is constant, the portfolio value changes over the interval by an amount

\[
V_{t_1} - V_{t_0} = h_{t_1} S_{t_1} - h_{t_0} S_{t_0} = h_t (S_{t_1} - S_{t_0}) = \int_{t_0}^{t_1} h_t dS_t,
\]

(C.1)

where the integral is defined trajectory-wise. Thus, we have the standard Itô value dynamics

\[
dV_t = h_t dS_t.
\]

**Wick-financing portfolio** According to Eq. (C.1), the buy-and-hold portfolio will satisfy

\[
V_{t_1} - V_{t_0} = h_{t_1} (S_{t_1} - S_{t_0}). \quad (C.2)
\]

However, to qualify as Wick-financing, the portfolio should instead satisfy the condition

\[
V_{t_1} - V_{t_0} = \int_{t_0}^{t_1} h_t S_t \circ dw^H_t, \quad (C.3)
\]

which generally differs from (C.2) since (C.3) is not generally consistent with

\[
h_t = \int_{t_0}^{t_1} S_t \circ dw^H_t,
\]

by non-commutativity of the Wick product i.e., \((F \circ G) \cdot H \neq F \circ (G \cdot H)\).

As an illustrative example, we construct a portfolio strategy that is self-financing in the standard sense but is not Wick-financing. To this end, we refer to an example in Björk and Hult (2005). We also demonstrate that the use of “risk-neutral” pricing formulas based on the Wick-financing concept, as suggested by Elliott and Van Der Hoek (2003), leads to easily implementable arbitrage possibilities in the standard naive sense.

**Example C.1 (Example 1 in Björk and Hult (2005))**

Consider the following portfolio strategy with initial capital \( x > 0 \).
1) At \( t = 0 \), we deposit all our money into a bank account and wait until \( t = 1 \).

2) Since the short rate equals zero, amount \( x \) remains in the account at \( t = 1 \).

3) At \( t = 1 \), we transfer all our money into a risky asset. Thus, we buy \( x/S_t \) shares at price \( S_t \) and hold this position until \( t = 2 \). Clearly, the value of our portfolio at \( t = 2 \) is given by

\[ V_2 = \frac{x}{S_t} \]

Since no capital has been added or withdrawn between \([0, 2]\), this portfolio strategy must be included in any reasonable definition of a self-financing portfolio. However, this strategy is not self-financing in the language of Elliott and Van Der Hoek (2003). To prove this result, we must show that

\[ x \frac{S_t^2}{S_1} \neq x + \int_0^2 h_n^1 S_t \circ dW_t^H. \]

In fact, even the expected values of the terms in the above expression are unequal. For the right-hand side, we have

\[ E \left( x + \int_0^2 h_n^1 S_t \circ dW_t^H \right) = x, \]

whereas for \( H \neq 1/2 \), the expected value of the left-hand side is

\[ E \left( x \frac{S_t^2}{S_1} \right) = x E \left( \exp \left\{ \frac{w_t^H - \frac{1}{2} 2^H}{2} \right\} \exp \left\{ \frac{w_t^H + \frac{1}{2} 2^H}{2} \right\} \right), \]

\[ = x E \left( \frac{1}{2} (2^H - 1) E \left( \exp \left\{ \frac{w_t^H - w_t^0}{2} \right\} \right) \right), \]

\[ = x E \left( \frac{1}{2} (2^H - 1) \exp \left\{ \frac{1}{2} (2^H - 1) \right\} \right), \]

\[ = x \exp \left\{ 1 - 2^{-2H} \right\} \neq x, \]

where \( S_t = s_0 \exp \left\{ w_t^H - \frac{1}{2} 2^{2H} \right\} \).

Appendix D. Dickey and Fuller Test

Let \( X_1, \ldots, X_n \) be observations from the AR(1) model

\[ X_t - \mu = \phi_1 (X_{t-1} - \mu) + Z_t, \quad Z_t \sim \text{iid}(0, \sigma^2), \quad (D.1) \]

where \( |\phi_1| < 1 \) and \( \mu = E[X_1] \). For large \( n \), the maximum likelihood estimator \( \hat{\phi}_1 \) of \( \phi_1 \) is approximately \( N(\phi_1, (1 - \phi_1^2)/n) \). If a unit root exists, this normal approximation is invalid (even asymptotically), so cannot be used to test the unit root hypothesis \( H_0 : \phi_1 = 1 \) vs. \( H_1 : \phi_1 < 1 \). To construct a test for \( H_0 \), we write the model (D.1) as

\[ \nabla X_t = X_t - X_{t-1} = \phi_0^* + \phi_1^* X_{t-1} + Z_t, \quad Z_t \sim \text{iid}(0, \sigma^2), \]

where \( \phi_0^* = \mu (1 - \phi_1) \) and \( \phi_1^* = \phi_1 - 1 \). Now let \( \hat{\phi}_1^* \) be the ordinary least squares estimator of \( \phi_1^* \) found by regressing \( \nabla X_t \) on 1 and \( X_{t-1} \). The estimated standard error of \( \hat{\phi}_1^* \) is

\[ \text{SE} \left( \hat{\phi}_1^* \right) = s \left( \frac{1}{n} (X_{t-1} - \bar{X})^2 \right)^{-1/2}, \]

where \( s^2 = \sum_{t=2}^n \left( \nabla X_t - \hat{\phi}_0^* - \hat{\phi}_1^* X_{t-1} \right)^2 / (n - 3) \) and \( \bar{X} \) is the sample mean of \( X_1, \ldots, X_{n-1} \). Dickey and Fuller derived the limit distribution of the t-ratio \( \hat{\phi}_1^* / \text{SE} \left( \hat{\phi}_1^* \right) \) as \( n \to \infty \) under the unit root assumption \( \phi_1^* = 0 \), which allows a test of the null hypothesis \( H_0 : \phi_1 = 1 \). The details are provided in Brockwell and Davis (2002).

References


