Original Paper — *Forma*, **27**, 37–43, 2012

Escher Degree of Non-periodic L-tilings by 2 Prototiles

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(Received February 15, 2012; Accepted April 27, 2012)

For a given tiling of the euclidean plane $\mathbf{E}^2$, we call the degree of freedom of perturbed edges of prototiles *escher degree*. In this paper we consider non-periodic L-tilings by 2 prototiles and obtain the escher degree of them.

Key words: L-tiling, Non-periodic hierarchical tiling, Escherization

1. Introduction

A non-periodic L-tiling is a limit of the sequence of patches as shown in Fig. 1. (Often this is called *a chair tiling*.) It is well known that this is a tiling of the euclidean plane $\mathbf{E}^2$ and that it has no periodicity of parallel translation. In [1], Sugihara introduces escherization of a plane tiling. Let $\mathcal{T}$ be a tiling of $\mathbf{E}^2$. If we have a finite set $\mathcal{S} = \{\alpha_1, \alpha_2, \cdots, \alpha_\ell\}$ of connected regions, and each tile of $\mathcal{T}$ is (orientation preserving) congruent to one of $\alpha_1, \alpha_2, \cdots, \alpha_\ell$, then we call $\mathcal{S}$ *the protoset* and $\alpha_1, \alpha_2, \cdots, \alpha_\ell$ *prototiles*. If we perturb some of edges of prototiles and get another tiling of the plane, we call the process of perturbation *escherization* of the tiling $\mathcal{T}$. This is a famous technique in artworks of M. C. Escher.

For example, see Fig. 2. The left figure is a tiling by one parallelogram. We can perturb the horizontal edges and slanted edges independently as in the right figure. In this paper, we determine the escher degree of L-tilings, that is, the degree of freedom of perturbed edges of prototiles of L-tilings. If the protoset of an L-tiling consists of one prototile, then the escher degree is one. This is shown in Theorem A. If the protoset of an L-tiling consists of two prototiles, then we show that there are 6 types of non-trivial tilings. (A non-trivial tiling is a tiling whose escher degree is more than 1.) This is shown in Theorem B and Theorem C.

In [2], Goodman-Strauss gives an aperiodic protoset derived from the L-tiling. We have new kinds of protosets also from the L-tiling in these theorems, but none of them are aperiodic. This is shown in Appendix B.

From the viewpoint of tiling rules, the escher degree may give a new method to observing equivalent classes of tilings with tiling rules and matching rules. About matching rules and substitution rules, see [3].

This paper is organized as follows. In Section 2 we prepare some notations and basic lemmas. In Section 3 we consider an L-tiling by one prototile. In Section 4, 5 we consider L-tilings by two prototiles. In Appendix, we show figures of tilings.

2. Preliminary

In this section, first we introduce a non-periodic hierarchical tiling. Let $\mathcal{S} = \{\alpha_1, \alpha_2, \cdots, \alpha_\ell\}$ be a set of connected regions and $\lambda > 1$ a constant. Let α'_i ($i = 1, 2, \cdots, \ell$) be a λ scale-up copy of α_i.

Suppose that each α'_i ($i = 1, 2, \cdots, \ell$) can be tiled by prototiles $\alpha_1, \cdots, \alpha_\ell$. That is, each α'_i can be divided into some of copies of $\alpha_1, \cdots, \alpha_\ell$. Let α''_i be a λ scale-up copy of α'_i. Then in the same way α''_i can be tiled by $\alpha'_1, \cdots, \alpha'_\ell$. Substituting $\alpha_1, \cdots, \alpha_\ell$ into $\alpha'_1, \cdots, \alpha'_\ell$, we have a tiling of α''_i by $\mathcal{S}$.

The tiling rules of α'_is by $\mathcal{S}$ are called *substitution rules*. A tiling of $\mathbf{E}^2$ obtained by substitution rules is called *a hierarchical tiling*, and if it doesn't have no periodicity of parallel translation, it is called *non-periodic*.

An L-tiling is an example of a non-periodic hierarchical tiling. Let $\alpha =$  and $\mathcal{S} = \{\alpha\}$. Let $\lambda = 2$ and $\alpha' =$ , then it gives a substitution rule. We call this tiling *an L-tiling* and this rule *L-substitution*.

DEFINITION 2.1 (s-SPREAD) *We call α', a patch of once L-substitution from α, 1-spread. We call $\alpha^{(s)}$, a patch of s times L-substitution from α, s-spread.*

Next, we define a edge and a perturbed edge.

DEFINITION 2.2 (EDGE) *Let an edge be a pair of a segment and a one-side neighborhood. See Fig. 3.*

For an edge, we regard a segment as a edge of a prototile, and one-side neighborhood as inside of a prototile. In the sequel, we do not distinguish an edge of a prototile from an edge in this definition. Next, we define a perturbed edge.

DEFINITION 2.3 (PERTURBED EDGE) *For an edge, fixing the both ends of the edge and perturbing it a little, we get a perturbed edge. See Fig. 4. For two perturbed edges a, b, $a = b$ if they are congruent.*

In order to perturb edges of prototiles, there exists restriction on a way of perturbing, because each prototile must be connected. In our context, we only concern degree of freedom of perturbed prototiles, so we consider only a per-

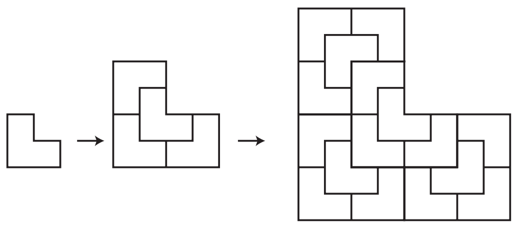


Fig. 1. Non-periodic L-tiling.

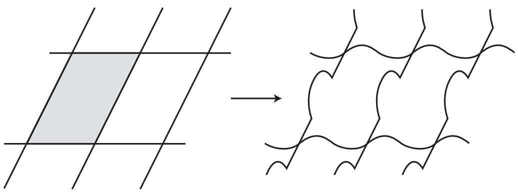


Fig. 2. Perturbation of edges of parallelogram.

Fig. 3. An edge.

Fig. 4. Examples of perturbed edges.

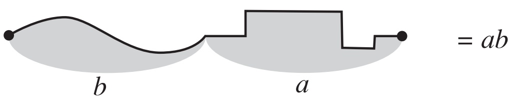


Fig. 5. An example of a product of perturbed edges.

turbed edge which is a little perturbed to avoid the restriction.

Next, we define a product of perturbed edges.

DEFINITION 2.4 (PRODUCT OF EDGES) *Let $a_1, a_2, \cdots, a_k$ be perturbed edges. If they are placed on a straight line from right to left and form a row, then we call it a product of $a_1, a_2, \cdots, a_k$ and we denote this product by $a_1 a_2 \cdots a_k$. See Fig. 5.*

For a perturbed edge a, we define two operations $\overline{a}$, and a^{-1}. For a perturbed edge a, $\overline{a}$ is a symmetry (right-side-left) image of a. In the same way, a^{-1} is an upside-down image of a. See Fig. 6.

It is easy to show the following lemma.

LEMMA 2.5 *(1)* $\overline{(\overline{a})} = a$, $(a^{-1})^{-1} = a$
(2) $\overline{ab} = \overline{a}\,\overline{b}$, $(ab)^{-1} = b^{-1}a^{-1}$
(3) $\overline{(a^{-1})} = (\overline{a})^{-1}$

In a tiling, if a tile with a perturbed edge a and another tile with a perturbed edge b are neighbors at a and b, we

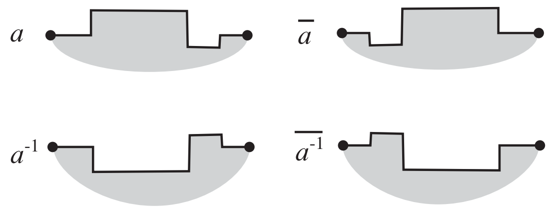


Fig. 6. Definition of $\overline{a}$, and a^{-1}.

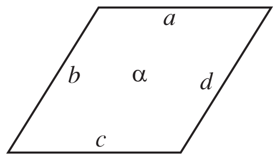


Fig. 7. Edges of the parallelogram α.

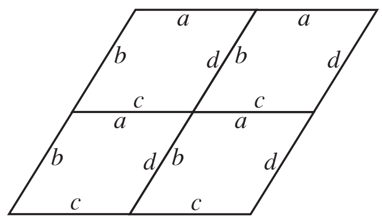


Fig. 8. Matching of the tiling (P1).

have $a = \overline{b^{-1}}$. We denote this relation by $\dfrac{a}{b}$. We often say that a matches b.

The following lemma is trivial.

LEMMA 2.6 *(1)* $\dfrac{a}{b}$ *if and only if* $\dfrac{b}{a}$
(2) If $\dfrac{a}{b}$ *and* $\dfrac{a}{c}$ *then* $b = c$
(3) $\dfrac{ab}{cd}$ *if and only if* $\dfrac{a}{d}$ *and* $\dfrac{b}{c}$

Let $\mathcal{T}$ be a tiling with respect to a protoset $\mathcal{S}$. Suppose that all prototiles are polygons. Here we assume that there is no vertex of a tile lying on an edge of another tile.

DEFINITION 2.7 (ESCHERIZATION, ESCHER DEGREE) *(1) Let $\mathcal{T}$ and $\mathcal{S}$ be as above. If we perturb edges of prototiles such that the perturbed prototiles give another tiling, we call this process escherization.*

(2) If the set of escherization of $\mathcal{T}$ is parametrized by some perturbed edges, the escher degree is the number of the parameters.

Example. Let α be a parallelogram and (P1) a tiling of $\mathbf{E}^2$ as in Fig. 2. Let a, b, c, d be edges of α as in Fig. 7.

From the matching of the tiling, we have $\dfrac{a}{c}$ and $\dfrac{b}{d}$. That is, if we perturb a, then the edge c changes such that $c = \overline{a^{-1}}$, and we can perturb b independently of a. Then the edge d changes such that $d = \overline{b^{-1}}$. See Fig. 8.

We call relations obtained from the tiling property *edge-matchings*.

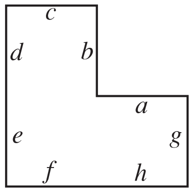


Fig. 9. Prototile α.

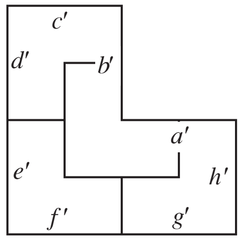


Fig. 11. Perturbed edges of α'.

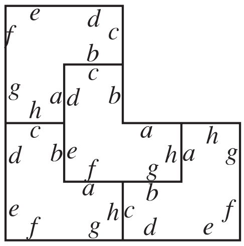


Fig. 10. Matching in 1-spread of α'.

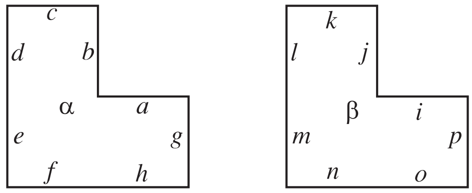


Fig. 12. Two prototiles α and β.

Hence all escherization of the tiling (P1) is parametrized by edges a and b. So the escher degree is 2.

3. Escher Degree of L-tiling by One Prototile

In this section we show that the escher degree of L-tiling by one prototile is one. We assume that an L-figure prototile α has 8 perturbed edges $a, b, c, \cdots, h$ as in Fig. 9.

THEOREM 3.1 (THEOREM A) *(1) In a non-periodic L-tiling by one prototile α, we have* $\dfrac{a=c=e=g}{b=d=f=h}$.

(2) The escher degree of this tiling is one. See Fig. 18.

REMARK 3.2 $\dfrac{a=c=e=g}{b=d=f=h}$ *means* $a=c=e=g$, $b=d=f=h$, *and* $\dfrac{a}{b}$

Proof: (1) Considering 1-spread, we directly have $\frac{b}{c}, \frac{a}{d}, \frac{h}{c}, \frac{b}{e}, \frac{f}{a}, \frac{h}{c}, \frac{g}{b}, \frac{h}{a}$ (see Fig. 10). This follows that $\dfrac{a=c=e=g}{b=d=f=h}$.

(2) The proof of (1) implies the following lemma.

LEMMA 3.3 *There exists a* 1*-spread of* α *if and only if* α *satisfies* $\dfrac{a=c=e=g}{b=d=f=h}$.

Let α' be a 1-spread and $a', b', \cdots, h'$ its edges (see Fig. 11).

Then we have $a' = ha$, $b' = bc$, $c' = e' = g' = de$, $d' = f' = h' = fg$ (see Fig. 10). If $\dfrac{a=c=e=g}{b=d=f=h}$ then we easily show that $\dfrac{a'=c'=e'=g'}{b'=d'=f'=h'}$. (For example, $h = d$ and $a = e$ implies $a' = ha = de = c'$, $\frac{h}{c}$ and $\frac{a}{b}$ implies $\frac{ha}{bc}$ and $\frac{a'}{b'}$.) This follows that a 1-spread of α' exists, that is, a 2-spread of α exists.

In the same way, if we have an $(s-1)$-spread $\alpha^{(s-1)}$ of α with edges $a^{(s-1)}$, $b^{(s-1)}$, $\cdots$, $h^{(s-1)}$, and there exists an s-spread, then it satisfies that $\dfrac{a^{(s-1)}=c^{(s-1)}=e^{(s-1)}=g^{(s-1)}}{b^{(s-1)}=d^{(s-1)}=f^{(s-1)}=h^{(s-1)}}$. If we set

$$\begin{aligned} a^{(s)} &= h^{(s-1)}a^{(s-1)}, \\ b^{(s)} &= b^{(s-1)}c^{(s-1)}, \\ c^{(s)} &= e^{(s)} = g^{(s)} = d^{(s-1)}e^{(s-1)}, \\ d^{(s)} &= f^{(s)} = h^{(s)} = f^{(s-1)}g^{(s-1)} \end{aligned}$$

inductively, then they satisfy $\dfrac{a^{(s)}=c^{(s)}=e^{(s)}=g^{(s)}}{b^{(s)}=d^{(s)}=f^{(s)}=h^{(s)}}$. and it follows that an 1-spread of $\alpha^{(s)}$ exists, that is, we have an $(s+1)$-spread of α.

If α satisfies $\dfrac{a=c=e=g}{b=d=f=h}$ then an s-spread $\alpha^{(s)}$ exists for any s. Hence the escherization is parametrized by a and the escher degree is one. This completes the proof.

4. L-tiling by Two Prototiles (1)

In this section, we consider the cases where two prototiles α, β (Fig. 12) make one 1-spread α'.

From Theorem A, if α' satisfies $\dfrac{a'=c'=e'=g'}{b'=d'=f'=h'}$ then α' has s-spread for $s = 1, 2, 3, \cdots$. We observe 5 patterns of α' in Fig. 13. We call them no.8, no.4, no.2, no.3, and no.5 respectively. (The numbering order is not ascending nor descending. These numberings are determined by the order of α and β.)

And we have the following theorem.

THEOREM 4.1 (THEOREM B) *(1) For no.2, no.3, no.4, the escher degree is* 1.

(2) For no.5, we have $\dfrac{a=e=m}{b=f=n}, \dfrac{c=g=i=k=o}{d=h=j=l=p}$ *and the escher degree is* 2 *(see Fig. 19).*

(3) For no.8, we have

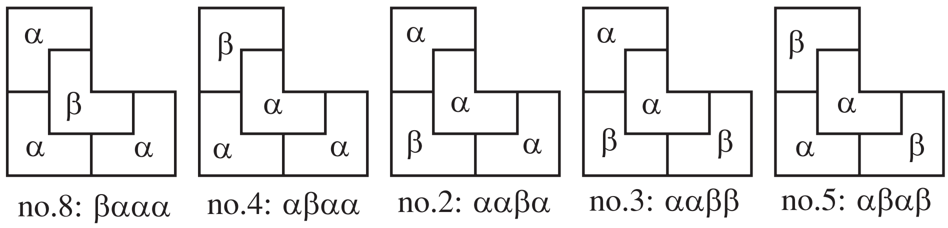


Fig. 13. Five patterns of 1-spread.

$\frac{a}{l=n=p}, \frac{k=m=o}{b}, \frac{c=g}{d=h}, \frac{e=i}{f=j}$ and the escher degree is 4 *(see Fig. 20).*

Proof: If α' satisfies $\frac{a'=c'=e'=g'}{b'=d'=f'=h'}$, then α' has n-spread for any n. So, it is sufficient to solve the edge-matching in α' and $\frac{a'=c'=e'=g'}{b'=d'=f'=h'}$.

For example, for no.5, edge-matching is given as in Fig. 14 and we have $\frac{j}{c}, \frac{i}{d}, \frac{p}{c}, \frac{b}{e}, \frac{f}{a}, \frac{h}{k}, \frac{g}{j}, \frac{h}{i}$. $\frac{a'=c'=e'=g'}{b'=d'=f'=h'}$ implies $\frac{pa=lm=de=lm}{bk=no=fg=no}$. We solve the system of equation and we have $\frac{a=e=m}{b=f=n}, \frac{c=g=i=k=o}{d=h=j=l=p}$. Inversely if $\frac{a=e=m}{b=f=n}, \frac{c=g=i=k=o}{d=h=j=l=p}$ then $\frac{a'=c'=e'=g'}{b'=d'=f'=h'}$ is satisfied. For other tilings, we can solve the system of relations in a similar way.

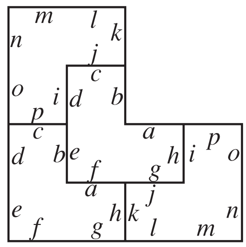


Fig. 14. α' for no.5 tiling.

REMARK 4.2 *If we have $\frac{a=c=e=g=i=k=m=o}{b=d=f=h=j=l=n=p}$, it follows that $\alpha=\beta$. So we can return the case of one prototile α and the escher degree is 1. This means that there are no solution of two distinct prototiles. In the two prototiles case, if the escher degree is more than* 1*, we call the tiling non-trivial. There are two types (no.5 and no.8) of non-trivial tilings by two prototiles and one* 1*-spread.*

5. L-tiling by Two Prototiles (2)

In this section, we consider two prototiles α, β (Fig. 12) and make two 1-spreads α', β'. There are 16 possibilities for α' and so as for β'.

We determine numbering for α' and β' as in Fig. 15, and we represent a tiling by a pair of two numbers for α' and β'. For example, (5, 10) means that α' and β' given in Fig. 16.

We remove the case $\alpha'=\beta'$ and two trivial cases (0, 15) and (15, 0), there remains 238 combinations. If we exchange the role of α and β, we know that (i, j) and $(15-j, 15-i)$ are equivalent. From the following lemma, we conclude that the number of remaining combinations is 119.

LEMMA 5.1 *$(i, 15-i)$ and $(15-i, i)$ are equivalent.*

Proof: If we denote $\alpha^{(s)}_{(i,j)}$ (resp. $\beta^{(s)}_{(i,j)}$) by the s-spread of α (resp. β) of tiling (i, j), it is easily show that $\alpha^{(2k-1)}_{(i,15-i)} = \beta^{(2k-1)}_{(15-i,i)}, \beta^{(2k-1)}_{(i,15-i)} = \alpha^{(2k-1)}_{(15-i,i)}, \alpha^{(2k)}_{(i,15-i)} = \alpha^{(2k)}_{(15-i,i)}, \beta^{(2k)}_{(i,15-i)} = \beta^{(2k)}_{(15-i,i)}$, for any $k=1,2,\cdots$. This completes the proof.

Here we have the third theorem.

THEOREM 5.2 (THEOREM C) *(1) If the escher degree of (i, j) tiling is more than* 1*, then $(i, j) =$ (5, 10), (10, 5), (0, 2), (13, 15), (0, 8), (7, 15), (0, 10), (5, 15).*

(2) For the tiling (5, 10) *(equivalently* (10, 5)*), $\frac{a=c=g=m}{f=j=l=p}, \frac{e=i=k=o}{b=d=h=n}$ and the escher degree is* 2 *(see Fig. 21).*

(3) For the tiling (0, 2) *(equivalently* (13, 15)*), $\frac{a=c=e=g=i=k}{b=d=f=h=j=p}$ and the escher degree is* 5 *(see Fig. 22).*

(4) For the tiling (0, 8) *(equivalently* (7, 15)*), $\frac{a=c=e=g=k=m=o}{b=d=f=h=l=n=p}$ and the escher degree is* 3 *(see Fig. 23).*

(5) For the tiling (0, 10) *(equivalently* (5, 15)*), $\frac{a=c=e=g=k=o}{b=d=f=h=l=p}, \frac{m}{j}, \frac{i}{n}$ and the escher degree is* 3 *(see Fig. 24).*

For each (i, j), we solve a system of equations of edge-matchings of s-spread ($s=1,2,\cdots$).

In some cases, only i (resp. only j) determines the result. For example, the following lemma holds.

LEMMA 5.3 *The escher degree of* $(1, j)$ *is* 1 *for any* j.

Proof: Assume that α' is no.1 (see Fig. 17). From the edge-matching of α', we get $\frac{c=e=k=i}{b=h}, \frac{a}{d=f}, \frac{g}{j}$. Edges of α' is given by $a'=pa, b'=bc, c'=e'=de, d'=f'=fg, g'=lm, h'=no$ and

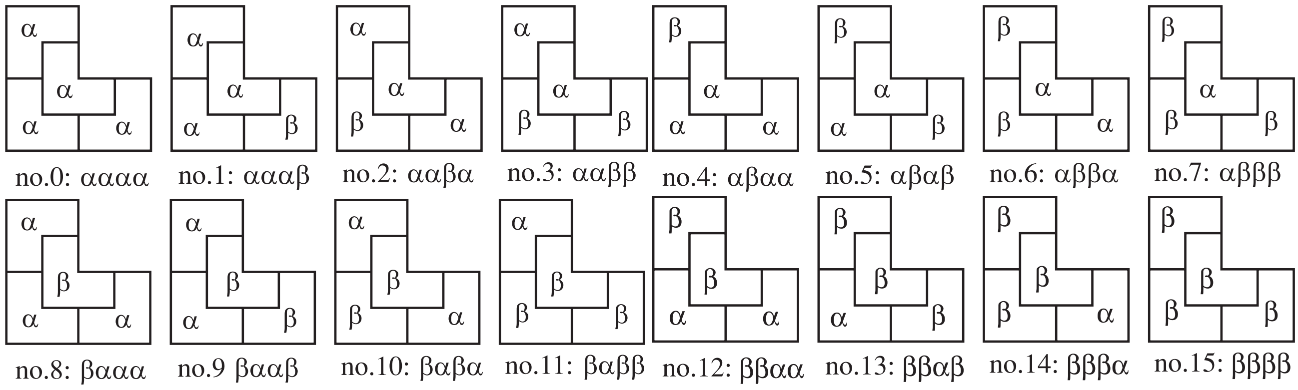


Fig. 15. Numbering for α' and β'.

we have $\frac{c'=e'}{b'=h'}, \frac{a'}{d'=f'}$, in the edge-matching of α'', and we get additional conditions $\frac{d}{c}, b = n, c = o, \frac{p}{g}$ and hence $\frac{a=c=e=k=i=o}{b=d=f=h=n}, \frac{g}{j=p}$. In the edge-matching in α''', we obtain another condition $\frac{d'}{c'}$, hence $\frac{d}{g}, \frac{e}{f}$, and we have $\frac{a=c=e=g=k=i=o}{b=d=f=h=j=n=p}$. In the edge-matching in α'''', we have $\frac{d'}{g'}$, hence $\frac{f}{m}$ and $\frac{g}{l}$, and we have $\frac{a=c=e=g=i=k=m=o}{b=d=f=h=j=l=n=p}$. This implies that within at most 4-spread all edges $a, b, \cdots, p$ are related and $\alpha = \beta$. This completes the proof.

And in a similar way, we can show that the escher degree of any tiling other than $(i, j) = (5, 10), (10, 5), (0, 2), (13, 15), (0, 8), (7, 15), (0, 10), (5, 15)$ is 1.

(2) In $(i, j) = (5, 10)$ case, the edge-matching of α' is $\frac{j}{c}, \frac{i}{d}, \frac{p}{c}, \frac{b}{e}, \frac{f}{a}, \frac{h}{k}, \frac{g}{j}, \frac{h}{i}$ (see the left of Fig. 16).

The edge-matching of β' is $\frac{b}{k}, \frac{a}{l}, \frac{h}{k}, \frac{j}{m}, \frac{n}{i}, \frac{p}{c}, \frac{o}{b}, \frac{p}{a}$ (see the right of Fig. 16). The edge-matchings of α', β' are equivalent to $\frac{a=c=g=m}{f=j=l=p}, \frac{e=i=k=o}{b=d=h=n}$. Let $a', b', \cdots, o', p'$ be edges of α', β' as in Fig. 11, we have

$$\begin{aligned}
&a' = pa, b' = bk, c' = lm, d' = no,\\
&e' = de, f' = fg, g' = lm, h' = no\\
&i' = hi, j' = jc, k' = de, l' = fg,\\
&m' = lm, n' = no, o' = de, p' = fg
\end{aligned}$$

Since the formula (c) holds for α', β', we have

$$\frac{a'=c'=g'=m'}{f'=j'=l'=p'}, \frac{e'=i'=k'=o'}{b'=d'=h'=n'}.$$

For example, $p = l$ and $a = m$ implies $a' = pa = lm = c'$, and so on. These are edge-matching of 2-spread α'', β''.

Inductively, we observe as follows. Suppose that we have $\alpha^{(s)}, \beta^{(s)}$. Let $a^{(s)}, b^{(s)}, \cdots, o^{(s)}, p^{(s)}$ be edges of $\alpha^{(s)}, \beta^{(s)}$.

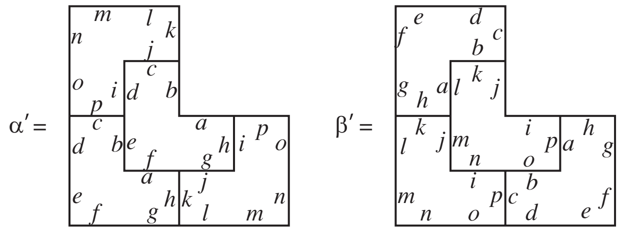
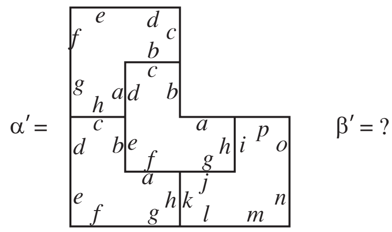


Fig. 16. Tiling (5, 10).



(1,j) case

Fig. 17. (1, j) tiling.

The relation between $a^{(s-1)}$s and $a^{(s)}$s are given by

$$\begin{aligned}
&a^{(s)} = p^{(s-1)}a^{(s-1)}, b^{(s)} = b^{(s-1)}k^{(s-1)},\\
&c^{(s)} = l^{(s-1)}m^{(s-1)}, d^{(s)} = n^{(s-1)}o^{(s-1)},\\
&e^{(s)} = d^{(s-1)}e^{(s-1)}, f^{(s)} = f^{(s-1)}g^{(s-1)},\\
&g^{(s)} = l^{(s-1)}m^{(s-1)}, h^{(s)} = n^{(s-1)}o^{(s-1)}\\
&i^{(s)} = h^{(s-1)}i^{(s-1)}, j^{(s)} = j^{(s-1)}c^{(s-1)},\\
&k^{(s)} = d^{(s-1)}e^{(s-1)}, l^{(s)} = f^{(s-1)}g^{(s-1)},\\
&m^{(s)} = l^{(s-1)}m^{(s-1)}, n^{(s)} = n^{(s-1)}o^{(s-1)},\\
&o^{(s)} = d^{(s-1)}e^{(s-1)}, p^{(s)} = f^{(s-1)}g^{(s-1)},
\end{aligned}$$

for any $s = 0, 1, 2, \cdots$. Using simple calculations, we have the following lemma.

LEMMA 5.4

If $\frac{a^{(s-1)} = c^{(s-1)} = g^{(s-1)} = m^{(s-1)}}{f^{(s-1)} = j^{(s-1)} = l^{(s-1)} = p^{(s-1)}}$,

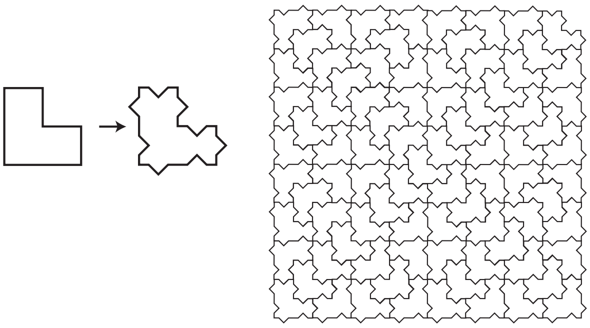


Fig. A.1. Tiling of one prototile.

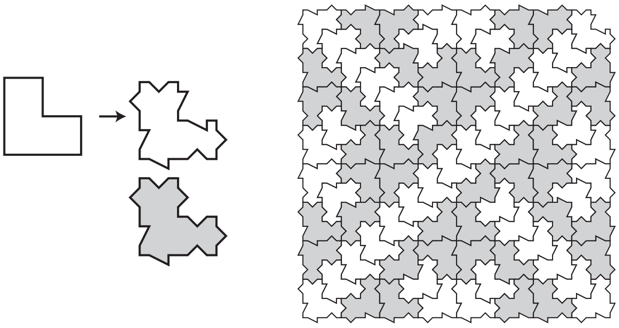


Fig. A.2. Tiling of two prototiles, no.5.

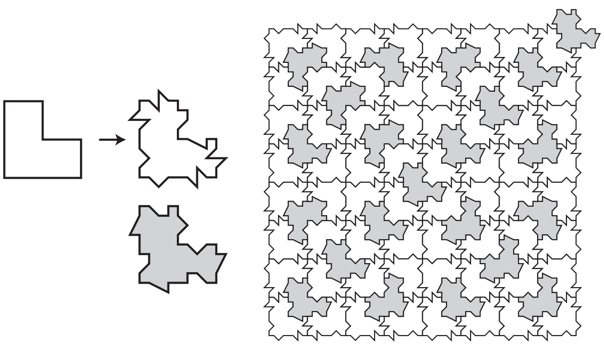


Fig. A.3. Tiling of two prototiles, no.8.

$\dfrac{e^{(s-1)} = i^{(s-1)} = k^{(s-1)} = o^{(s-1)}}{b^{(s-1)} = d^{(s-1)} = h^{(s-1)} = n^{(s-1)}}$, *then*
$\dfrac{a^{(s)} = c^{(s)} = g^{(s)} = m^{(s)}}{f^{(s)} = j^{(s)} = l^{(s)} = p^{(s)}}, \dfrac{e^{(s)} = i^{(s)} = k^{(s)} = o^{(s)}}{b^{(s)} = d^{(s)} = h^{(s)} = n^{(s)}}.$

Proof: For example, $p^{(s-1)} = l^{(s-1)}$ and $a^{(s-1)} = m^{(s-1)}$ implies $a^{(s)} = p^{(s-1)}a^{(s-1)} = l^{(s-1)}m^{(s-1)} = c^{(s)}$. Other relations are shown in a similar way. From this lemma, $(s+1)$-spreads $\alpha^{(s+1)}$, $\beta^{(s+1)}$ exist for any s.

(3), (4), and (5) are shown in a similar way as (2).

REMARK 5.5 *In Appendix, we show figures of these tilings. In* $(0, 2)$, $(0, 8)$ *tilings,* $\beta^{(s)}$ *contains only one tile* β. *This means that these tilings are equivalent to a tiling of* α *as tilings.*

Acknowledgments. The authos would like to thank Prof Koukichi Sugihara for his giving them such an interesting topics. The authors also thank the refree for his kind and useful advice.

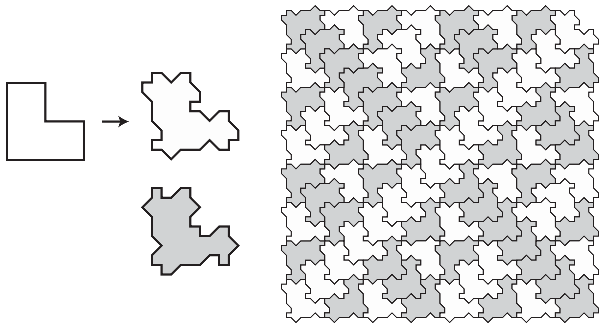


Fig. A.4. Tiling of two prototiles (5,10).

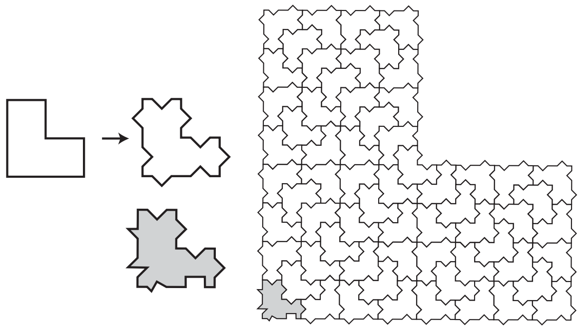


Fig. A.5. Tiling of two prototiles (0,2).

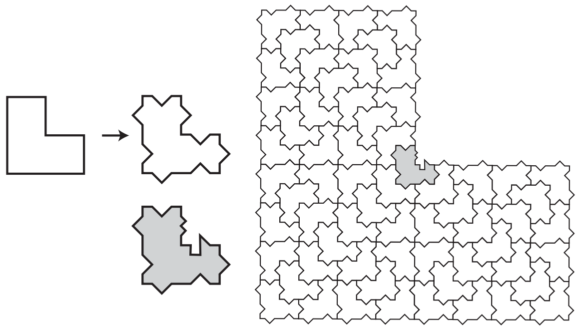


Fig. A.6. Tiling of two prototiles (0,8).

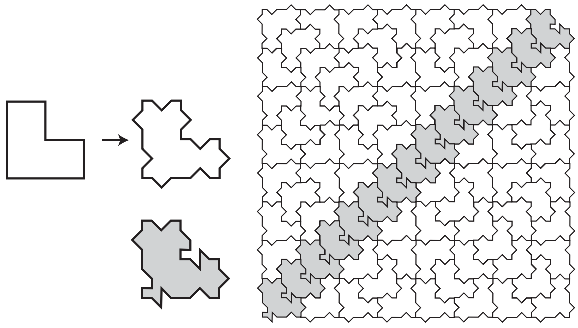


Fig. A.7. Tiling of two prototiles (0,10).

Appendix A.

From Figs. A.1 to A.7 are pictures of tilings appearing in Theorems A, B, and C.

Appendix B.

For a protoset $\mathcal{S}$, if **any** tiling by $\mathcal{S}$ has no periodicity

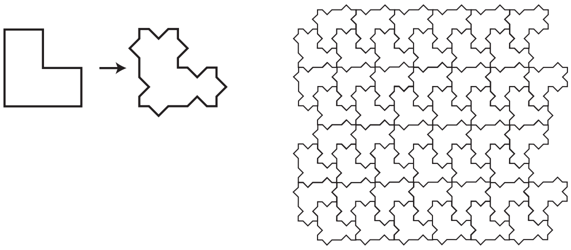


Fig. B.1. Periodic tiling of one prototile.

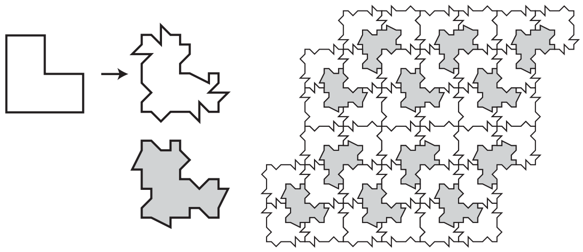


Fig. B.2. Periodic tiling of two prototiles no.8.

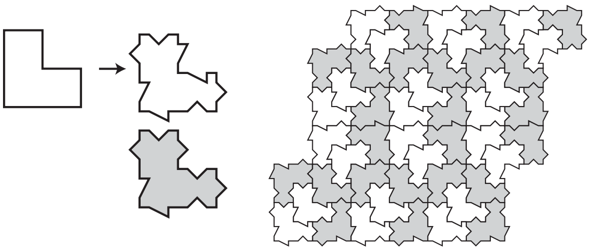


Fig. B.3. Periodic tiling of two prototiles no.5.

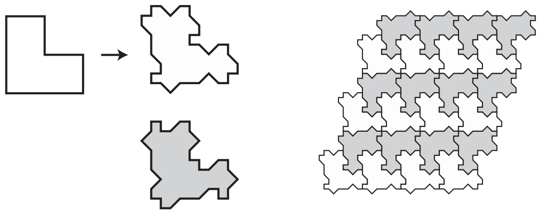


Fig. B.4. Periodic tiling of two prototiles (5,10).

of parallel translation then we call $\mathcal{S}$ *an aperiodic protoset*. Any tilings we obtain in this paper are not aperiodic protosets. From Figs. B.1 to B.4 are figures of periodic tilings.

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