Original Paper *Forma*, **19**, 265–277, 2004

The Role of Curie Principle in Understanding Composite Plane Symmetry Patterns: New Ethnomathematic Relations in Ancient Eurasian Ornamental Arts from Archaeologic Finds of the Period 1 M.B.C. and 1 M.A.D.

Szaniszló BÉRCZI

Department of General Physics, Eötvös University, H-1117 Budapest, Pázmány Péter sétány 1/a., Hungary
E-mail address: bercziszani@ludens.elte.hu

(Received June 5, 2004; Accepted October 8, 2004)

Keywords: Curie Principle, Eurasian Ornament, Composite Plane-Symmetry Patterns, Ethnomathematics

Abstract. Both architectural remnants (murals, tiles) and archaeologic finds from tomb excavations (textiles, bones, bronze mounts) exhibit various complex ornamental plane symmetry structures. Curie-principle may serve as a mathematical tool to reconstruct and classify these patterns. This classification of composite patterns may serve as a tool for comparative ethnomathematical studies in Eurasian ornamental arts, because the composite plane symmetry patterns can be considered as intuitive mathematical discoveries of ancient peoples. The survival of pattern structures were found in greater regions in the interval between 1 milleneum B.C. and 1 milleneum A.D. There is one with dominance of cm-cmm-pg-p4m-types in the Pontusian Steppe - Caucasus - Caspian region during the whole interval studied. There is another one in Central Asia in the middle of the 1 milleneum A.D. The third one is dominated by the p4m-p4-p2-types in the Cretean and Greek region.

1. Introduction

After frieze and double frieze patterns we studied plane symmetry groups (BÉRCZI, 1989) we improved our earlier studies in search of plane symmetry patterns of Eurasian ornamental arts. It was found (BÉRCZI, 2001) that many complex patterns can not be truly classified into the classical types. These patterns exhibited a type of complexity which can be analysed on the basis of addition (superposition) of the subsets of the pattern. This means that we could separate the complex plane symmetry patterns and divided them into simple patterns. These simple patterns are fitting on the same grid and scale. Their recombination (superposition) showed that the structure of the original complex pattern can be given in a sub-group style parametrisation.

The mathematical background to these ornamental structures contains the symmetries of the wallpaper patterns and their addition involves a principle of the interaction of symmetries: the Curie principle. The Curie principle identifies symmetries of the environment

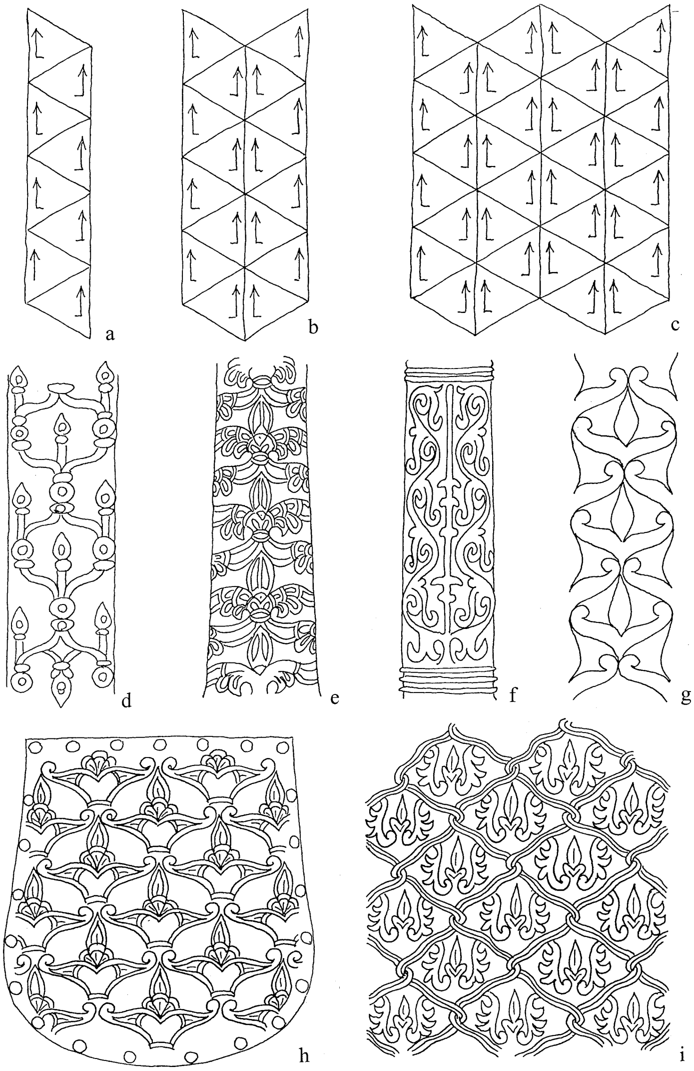


Fig. 1. The system of construction from a simple frieze pattern to a double frieze pattern and a plane symmetry pattern. Examples are from the Pontusian Steppe and Carpathian Basin: a) skeletal g-type frieze, b) skeletal mirror-doubled double frieze, c) extension of the patterns on a) and b) to form cm-type plane symmetry pattern, d) m-g type double frieze on the handle of a Scythian sword (Kelermes, 5 C.B.C.), e) m-g type double frieze on the handle of a sword (Kiev, 8–9 C.A.D.), f) m-g type double frieze on the handle of a sword (Warsawa, 8–9 C.A.D.), g) m-g type double frieze on a belt buckle from Hungary, h) cm-type pattern on a sabretache plate from Eperjeske, Hungary (8–9 C.A.D.), i) cm-type column ornament from a cathedral doorway from Gyulafehérvár, Hungary (12 C.A.D.).

as a constraint to the symmetries of a structure embedded into this environment. Applying Cuire principle to the complex patterns we find that the two (or more) patterns (to be superposed) form mutual constraints for each other. As a result of this interaction of the superposed patterns (they can be considered as layers), the initial plane symmety patterns reduce the symmetries of each other, and after superposition they have a final, generally reduced complex symmetry pattern.

Composite plane symmetry patterns may help in tracing the way of streaming of cultural products of peoples in the Eurasian regions in space and time (Prince MIKASA, 1994). In the second part of the paper these complex patterns were used to sketch regions of their distribution in Eurasia. These regions can be considered as ethnomathematical regions of the ancient Eurasian Art. We focused on the Northern part of Eurasia and we did not deal with the Islamic arts, a main part of the Southern Eurasian art heritage, because they are well studied and classified yet and only the last centuries of our period brings the islamic art distribution to the regions we studied.

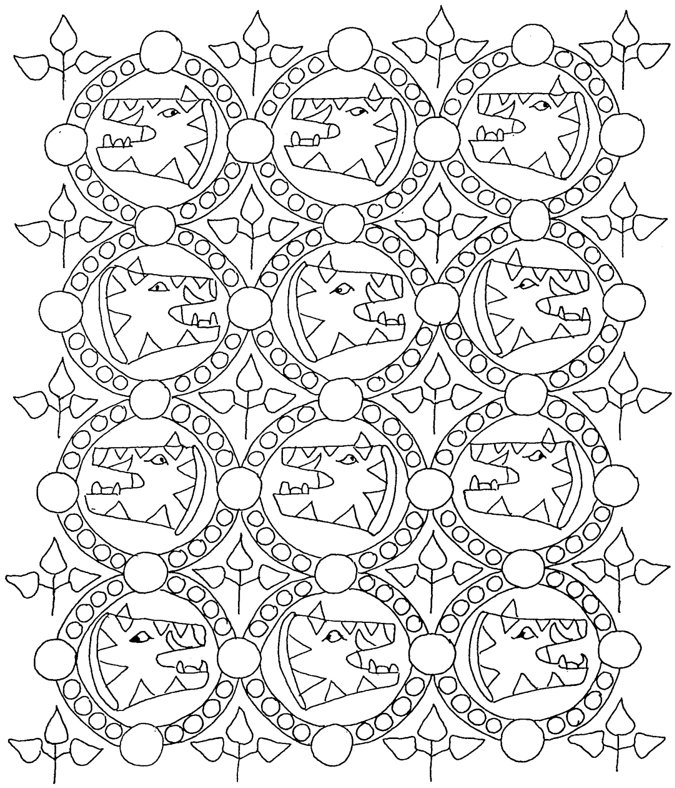
2. Method Study I: One More Hierarchy Level over Repeating Elements, Simple Structures: Composite Plane Symmetry Patterns

In the case of a simple structure with a given symmetry there are two system levels in the pattern: one is the repeating element, the other is the structure. The thing that they are

Fig. 2. Composite ornamental patterns on the dresses of the noble visitors in a party on a detail of murals of Afrasiab, Central Asia, Uzbeghistan (6–7 C.A.D.).

independent components can be seen easily from the fact that they can be separated or one of them can be replaced. If structure is fixed the element can be replaced (i.e. roses instead of trees). If the elements are fixed, the structure can be replaced (i.e. m type pattern instead of g). If the invariant part is the structure, we can identify some cultural relations on the basis of repeting elements. In cultural context a motif can be characteristic feature of friezes, populating all the frieze structures like palmette motif of Conquesting Hungarians in the Árpád Age (9–10 C.A.D.) in Hungary.

In the case of the composite patterns there are more hierarchy levels over element and structure. This level is the complex pattern. The most simple patterns are the frieze symmetries along the line. We could build complex (composite) patterns from these simple frieze types: the double frieze patterns. For double frieze patterns the simple friezes served



a

Fig. 3. a) Composite plane-symmetry pattern on the dress of the left visitor from the Afrasiab mural. b) GIS-like representation of the various pattern layers of this composite ornamental pattern. The uppermost layer exhibits the reference system, then the p4m-type layer comes with circles and beads, then the pm-type layer comes with the leaves and finally the pg-type pattern of the boar heads fit into the circles. After superposition of the layers the complex pattern exhibits a p4m/pm/pg structure.

as units and we developed the construction table of double friezes in a form of a matrix. Not all these double frieze patterns were represented in any rich ornamental arts of communities with intuitive mathematical discoveries. However, many of them were used in the Avarian, Onogurian, Old-Hungarian and Celtic arts (BÉRCZI, 2000) (Fig. 1).

In the case of the composite plane symmetry patterns there are also more hierarchy levels over element and structure. We may arrange them as follows: over elements the structure is a simple plane symmetry pattern. Over them we find the complex pattern: the complex (composite) plane symmetry pattern. Our method will be similar, in many respects, to that of our one in the case of double frieze patterns. There the basic friezes were the units, here we use simple plane symmetry patterns as units. We imagine that they were placed on transparent sheets (for example on projector transparencies) and we shall add them, that is we shall superpose them. Superposition will give the final complex plane symmetry pattern.

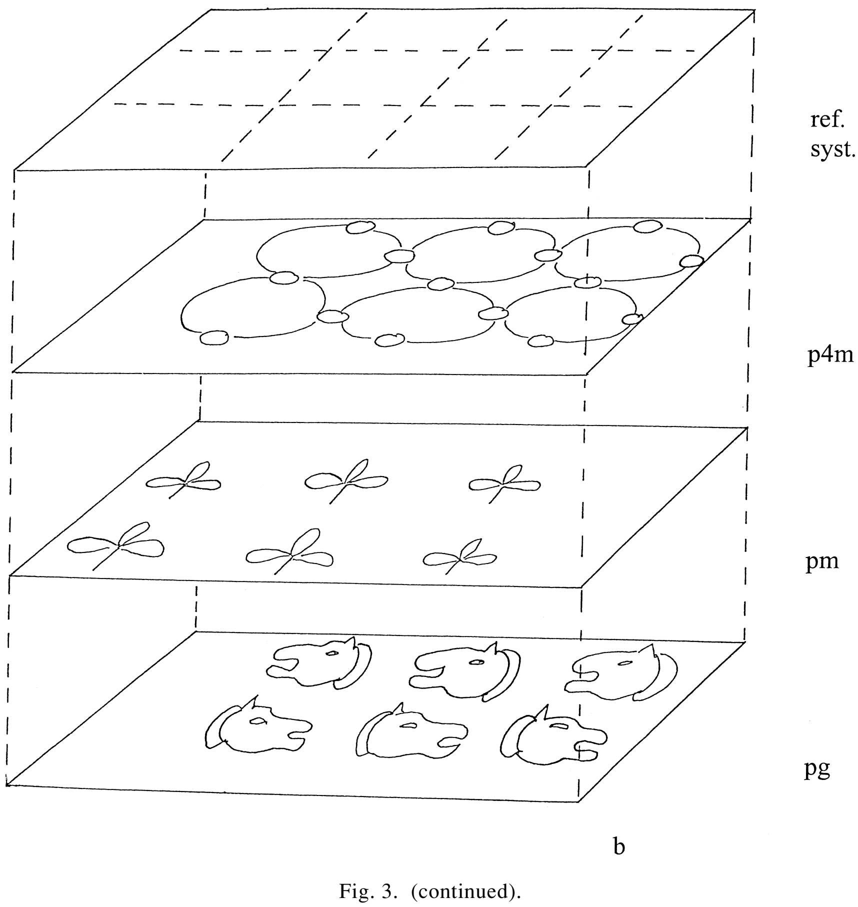


Fig. 3. (continued).

If we can recognize that a plane symmetry pattern can be separated into two such subsystems of patterns both of which alone form a pattern, then the initial pattern is called by us: complex composite pattern (in the future only composite pattern). The two separated patterns are called subsystem patterns. The most frequently occurring composite patterns are those where one subsystem pattern is formed from a net. First we show some patterns and we use in their description the subgroup structure showed in details about the colored symmetry (COXETER, 1986).

However, we may extend this definition to not only two but three or more subsystems, too. If we can recognize that a plane symmetry pattern can be separated into two or more such subsystems of patterns both of which alone form a pattern, then the initial pattern is also called: composite pattern.

3. Example Study I: Dresses of 3 Visitors from Afrasiab Murals, Medieval Samarkand, Central Asia

Only one small part of a mural, excavated in the medieval Afrasiab, exhibits several examples of the composite plane symmetry patterns. 3 visitors bring gifts for the princeps and the dresses of these visitors has composite patterns. Let us study them (Fig. 2). The left side visitor has a dress pattern composed from 3 layers. We begin from the higher symmetry layers. The first layer is the pattern of the double circles forming a net for the figural elements: this has a p4m symmetry. In the second layer in the smaller spaces between the double circles (concave regions) there are leaves, arranged in a pm type pattern. Finally the boar-heads form a pg type symmetry pattern.

For the most visual arrangement we use the modern GIS-like projection where multiple layers of data are placed into fitted (corresponding) database layers. We may show them one above the other accordingly: the p4m bead-circles with dots at connecting points (1st layer), the pm type leaves (2nd layer) and the pg type boar-heads (3rd layer) (Fig. 3).

On the mural the dress pattern of the left visitor has the same type of pattern shown on the textile in the hands of the middle man. The dress of this central man has more simple composite pattern. It consists of two layers. The more simple one is the (probably, not well visible) mirror symmetric hills between the birds. The hills have pm type pattern, while the birds form a pg pattern.

The dress of the right man has 3 layers of composite pattern. The net forms a cmm type pattern, the heart-like formed decorations at the top and bottom of the net-cells have pmg type symmetry and the senmurvs form a pg type pattern.

As a summary, patterns in this detail of the Afrasiab mural 5 different plane symmetry patterns occur as subsets: p4m, cmm, pmg, pm and pg. Although higher symmetries occur on the dresses, the final symmetry of these patterns is pg, because this has the lowest degree when patterns are arranged in a subset sequence. Curie principle determines this relation between the plane symmetry patterns forming a composite pattern.

4. Method Study II: The Composite Patterns and the Curie Principle

In our composite patterns at least two subsystem patterns form interacting. According to our GIS-like projection arrangement these interacting patterns are copied on a sheet transparent film and they can be superposed one above the other (Fig. 3). So their interaction is their superposition (unification), which results in that they mutually affect each other: therefore the Curie principle can be used to them. The Curie principle asserts that in superposition (addition) the two (or more) subsystems the symmetry group of the lowest order subsystem determines the symmetry group of the whole pattern (SHUBNIKOV and KOPTSIK, 1977).

When the two (or more) subsystems are superposed on each other they form constraints for each other. As a result of the superposition the final symmetry of the composite pattern can be determined: it can be formulated from the relation of the symmetry groups of the two (or more) component subsystem patterns. The product of this superposition is in strong relation with the results of the coloring of a pattern with a given symmetry group (COXETER, 1986). In order to use this formalism we give the correspondence between the two (or more) types of structures. There (COXETER, 1986) the symmetry of a 2-colored pattern was written in a form, where G noted the group of the uncolored pattern and G1 was the group of the colored pattern. G/G1 noted the type of the pattern, and this meant the order of the coloring, too. (The possibilities were given in the table I. of COXETER, 1986.)

In our case, the composite pattern contains a pattern with higher order of symmetry (with symmetry group G) and its symmetry is constrained (restricted) by a pattern (superposed on it) with a lower (G1) symmetry group. (In the case of more then 2 patterns in superposition they can be arranged into a sequence according to the gradually increasing degree of their symmetry groups.) The quotient group of G/G1 gives the symmetry group of the composite pattern according to the normal subgroup relations of the 17 plane crystallographic groups. The colouring of a composite plane symmetry pattern is possible according to these subgroup relations (COXETER, 1986; MACGILLAVRY, 1976). For our studies, however, the subgroup relations are enough, and therefore the composite wallpaper patterns are also “halfway” between plane symmetry groups and coloured plane symmetry groups (similarly as double threads were somewhere halfway between friezes and wallpaper patterns, in BÉRCZI, 1989). According to this composite pattern description system the patterns on the dresses of the 3 visitors of the Afrasiab mural has the following subset structure: left man: p4m/pm/pg, the central man: pm/pg, the right man: cmm/pmg/pg.

5. Example Study II: Composite Patterns from More Ancient Times in the Central Asian Neighbourhood

5.1. Royal Scythian Caucasian-region, Hun-Scythian Altai-region, Urartu

A gold plate from the excavations at Ziwiyeh has a composite pattern (see figure 14 of BÉRCZI, 2001a) It can be separated to two subsystem layers. One is a net of gentle palmette stems forming a tendrill with cm (almost cmm) symmetry group of this subsystem pattern. The second is a fill of the cells with animals, goats and deers, alternately changing in their rows. The animals have a simple p1 pattern, so after superpostion of the two subsystems all higher symmetry of the net pattern subsystem is violated, reduced to a p1 symmetry of the whole cm/p1 pattern of Ziwiyeh from the Caucasian-Scythian Iron Age (7 C.B.C.). (The cm type net was rather frequently occurring in the steppe belt of Eurasia, and we shall see that this pattern probably goes back to Urartu, BÉRCZI, 2001a.)

A shabrack from the kurgan excavations at Pazyryk has an applicated pattern

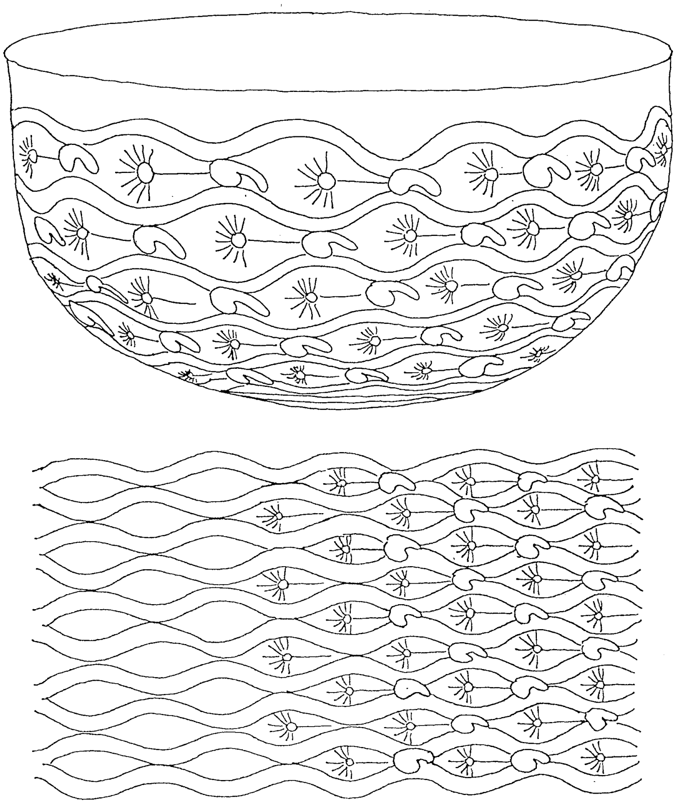


Fig. 4. Composite ornamental pattern on a cup from Crete with cmm/cm type pattern (regardless of dots).

(RUDENKO, 1953; BÉRCZI, 2001a). It has composite pattern with 3 separable subsystem layers. One layer is a square net pattern with p4m symmetry group. The second layer is formed by the circular ornaments placed to every second square (in a chess-table style: say whites) so they form a modified p4m′ pattern. The third layer is a symmetry subsystem pattern with horsehead figures placed into the free squares (in a chess-table style: now they are the blacks). This horseheads pattern has cm type plane symmetry group. Superposition of the three subsystem patterns results in reduction of all the mirror reflections of the p4m and p4m′ except that of which is in coincidence with the mirror reflections of cm. So the final structure of the composite pattern of the Pazyryk shabrack is: p4m/p4m′/cm (5 C.B.C.).

On a pavement tiling from the excavations at Nimrud palace there was found a mosaic composite pattern (GHIRSHMAN, 1964). The first layer of the pattern exhibits a p4 type net subsystem pattern and the second layer is a p4m type rosette subsystem pattern. In the superposition of the two subsystems the mirror symmetries of the p4m rosette pattern is violated by the p4 rotations only. Therefore the final composite symmetry of the pattern is: p4m/p4.

5.2. Cretean composite patterns

There are early occurrences of composite patterns in Crete: in Orchomenos, Knossos and Herakleion (15 C.B.C.). In the composite pattern of Orchomenos S-edged network form the first layer (p4) and leaves (p2) placed in the free spaces in the cells form the second layer with p4/p2 composite structure. In the pattern from the Knossos Palace also S-edged cells are forming the first layer (p4) and small rosette flowers sitting in the free spaces of the cells (p4m) form the other layer with p4m/p4 composite structure. In the composite pattern of Herakleion Museum 3 layers can be separated. The first is a gentle weaving rope, forming a tendrill of ropes with cmm symmetry group. The second layer is formed by a teeth-like motif placed in every second cells to form pm pattern. The third layer fills the free cells with a violated p4 rotational form, reduced to 2-fold rotation, so that the symmetry of this subsystem pattern is p2. Finally the whole system has cmm/pm/p2 composite pattern (BÉRCZI, 2001b). And there is a cup from Crete with sinusoid lines forming a first layer with cmm net, there are plants placed into the cells of the net with cm symmetry pattern, and finally asymmetric dots are palced on the knots of the net. Regardless of the dots the cup has beautiful cmm/cm composite pattern (Fig. 4).

6. Method Study III: Subgroup Relations for 12 Plane Symmetry Groups (without 3 rotations)

There are subgroup relations between the plane symmetry groups summarized by Coxeter (1986). They form a relation table where our archaeological findings can be projected to.

Table 1. Relation table between the subset of the plane symmetry groups (p3, p3m1, p31m, p6 and p6m types are missing here). This row/column subset relations table contains X at subgroup relations which occur in Eurasian arts. Numbers show the normal subgroup order relations.

	p1	p2	pg	pm	cm	pgg	pmg	pmm	cmm	p4	p4g	p4m
p1	2											
p2	2	2										
pg	2		2									
pm	2	**X**	2 **X**	2	2							
cm	2 **X**		2	2								
pgg	4	2	2									
pmg	4	2 **X**	2 **X**	2		2	2					
pmm	4	2	4	2	4	4	2	2	2			
cmm	4	2	4	4 **X**	2 **X**	2	2 **X**	2				
p4	4	2 **X**								2		
p4g	8	4				2		4	2	2		
p4m	8	4			**X**	4		2	2 **X**	2 **X**	2	2

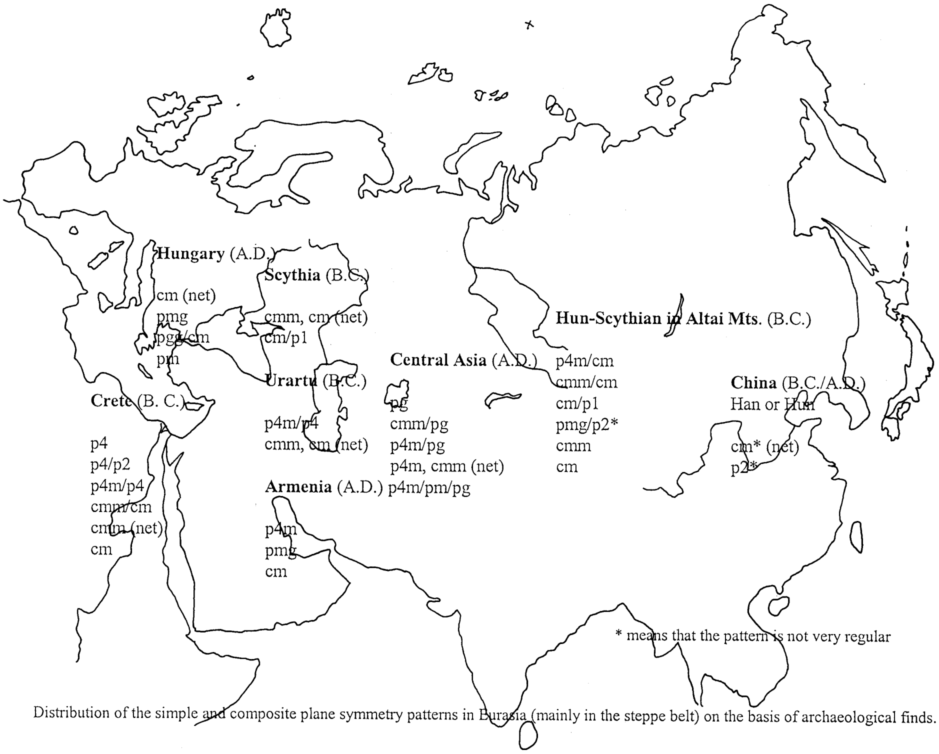


Fig. 5. Ethnomathematical regions with characteristic plane symmetry types in Eurasia.

7. Summary

Complex composite patterns were found in Eurasian arts from the Northern Eurasian greater regions in the interval between 1 milleneum B.C. and 1 milleneum A.D. (Fig. 5). They were architectural remnants and archaeologic finds. The Curie-principle was used as a mathematical tool in reconstructing and classifying these patterns. After classification of the composite patterns they were used in sketching ethnomathematical type regions in Northern Eurasia in the interval between 1 milleneum B.C. and 1 milleneum A.D. The Pontusian Steppe - Caucasus - Caspian region used cm-cmm-pg-p4m-types during the whole interval studied. The Central Asian arts were dominated by the cmm-pg types in the middle of the 1 milleneum A.D. The Cretean-Greek region was dominated by the p4m-p4-p2-types (Fig. 6). Inner-Asian Hun-Scythian art was characterised by the p4m-cmm-cm-types with interesting example of pmg/p2 case (Fig. 7).

REFERENCES

BÉRCZI, SZ. (1986) Escherian and non-escherian developments of new frieze types in Hanti and old Hungarian

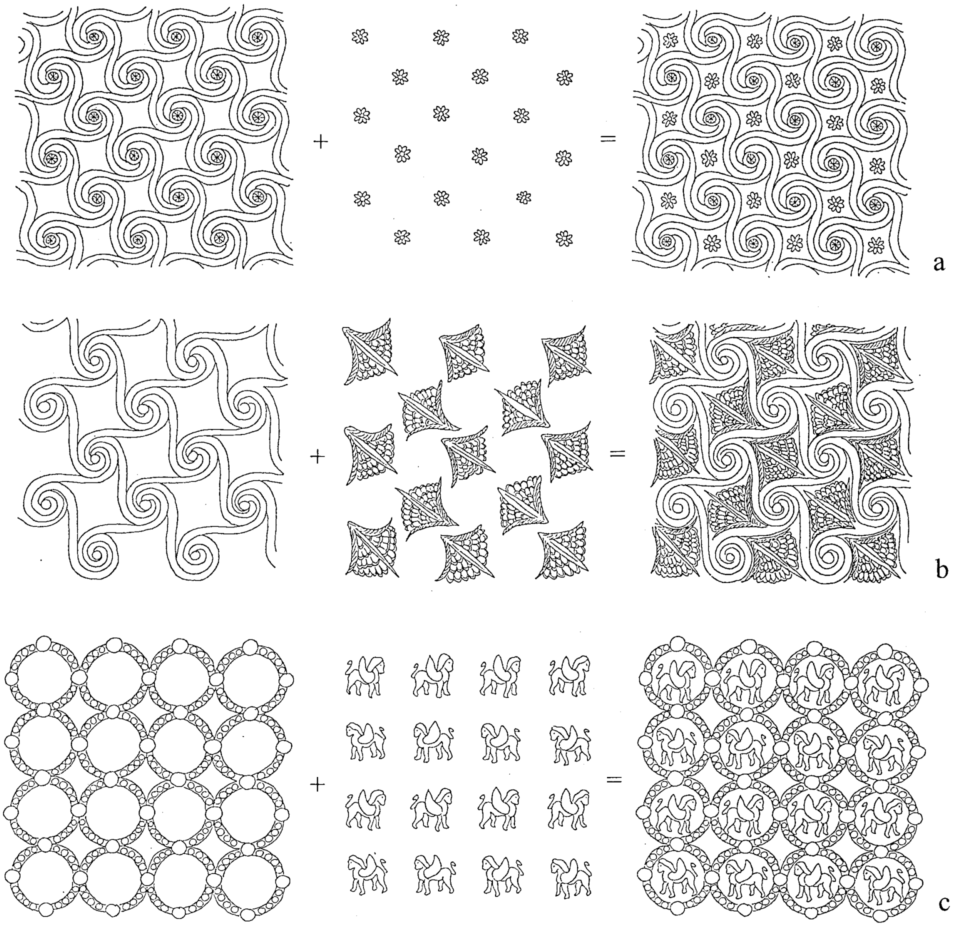


Fig. 6. Forming 3 composite pattern by addition of their simple plane symmetry subset patterns: a) upper row: pattern from Knossos (p4m/p2), b) middle row: pattern from Orchomenos (p4/p2), c) bottom row: pattern from Afrasiab, Central Asia (p4m/pg).

communal art, in *M. C. Escher: Art and Science* (eds. H. S. M. Coxeter *et al.*), pp. 349–358, North-Holland, Amsterdam.

BÉRCZI, Sz. (1987a) Szimmetriajegyek a honfoglaláskori palmettás és az avarkori griffes-indás díszítömüvészetben (Symmetry marks in the palmette art of the conquesting Hungarian times and the griffin-and-tendrill art of the Avarian times), *Cumania*, **10** (Bács-Kiskun Megyei Múzeumi Évkönyv, Annales of the Museums of Bács-Kiskun County, Hungary), 9–60 (in Hungarian).

BÉRCZI, Sz. (1987b) A szimmetria a régészetben (Symmetry in archaeology), *Tudomány* (Hungarian Edition of Scientific American), **1987**, February, 38–41 (in Hungarian).

BÉRCZI, Sz. (1987c) *Szimmetria és techné a magyar, avar és hanti díszítömüvészetben* (*Symmetry and Techné in the Magyar, Avarian and Chanti Ornamental Arts*), Leuveni Katolikus Egyetem, Collegium Hungaricum (Katalógus a kiállításokhoz, Catalogue for exhibitions), Leuven, 59 pp. (in Hungarian).

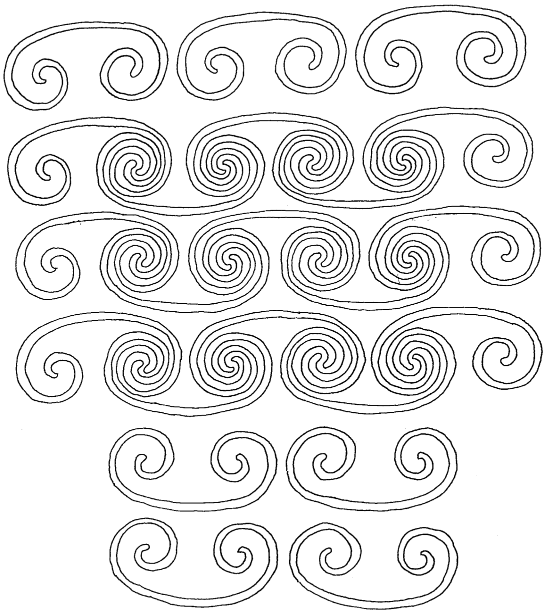


Fig. 7. Reconstruction of the idealised (corrected) complex pattern of the curtain found in Noin Ula, Mongolia. The pmg/p2 pattern is from the 1 C.B.C.–1 C.A.D. period, probably a Hun-Scythian heritage. a) the simple plane symmetry pattern pmg is the basis, where: b) the S-forms of p2 pattern were added. (The original pattern is not such regular.)

BÉRCZI, Sz. (1989) Symmetry and technology in ornamental art of old Hungarians and Avar-Onogurians from the archaeological finds of the Carpathian basin, seventh to tenth century A.D., in *Symmetry 2. Computers Math. Applic. CAMWA* (ed. I. Hargittai), **17**, No. 4–6, pp. 715–730, Pergamon Press, Oxford.

BÉRCZI, Sz. (1990) *Szimmetria és Struktúraépítés* (*Symmetry and Structure Building*), Lecture Note Series Book, **J3-1441**, Tankönyvkiadó, Budapest, 260 pp. (in Hungarian).

BÉRCZI, Sz. (2000) Katachi U Symmetry in the ornamental art of the last thousands years of Eurasia, *FORMA*, **15**/1, 11–28, Tokyo.

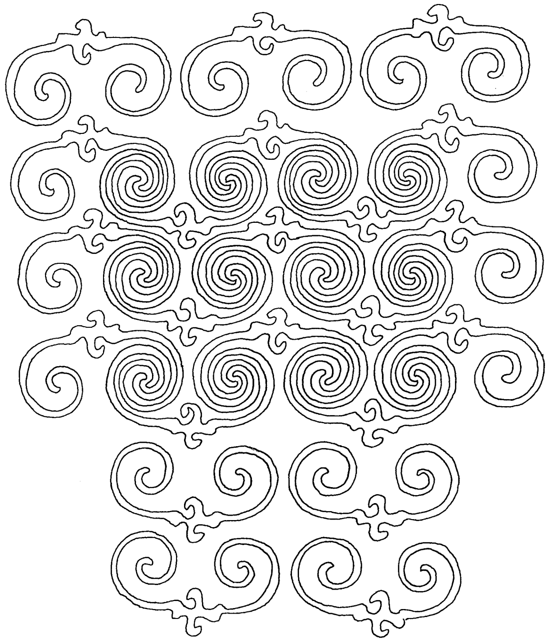
BÉRCZI, Sz. (2001a) On the earlier occurrences of the cm-type nets and wallpaper patterns in ornamental arts of central Eurasia, *Symmetry: Culture and Science*, **12**, No. 1–2, 5–24.

BÉRCZI, Sz. (2001b) Composite wallpaper patterns among intuitive ethnomathematical discoveries of the ancient Crete and Urartu, *Symmetry: Culture and Science*, **12**, No. 1–2, 39–52.

BÉRCZI, Sz., BÉRCZI, K., BÉRCZI, Z. S. and KABAI, S. (2001) Coloring booklet series about the arts of central Eurasia with rich heritage of ethnomathematical ornamental arts, *Symmetry: Culture and Science*, **12**, No. 1–2, 231–239.

COXETER, H. S. M. (1985a) A simple introduction to colored symmetry, *Internat. Journal of Quantum Chemistry*, **31**, 455–461.

COXETER, H. S. M. (1985b) The seventeen black and white frieze types, *C. R. Math. Acad. Sci. Canada*, **7.5**, 327–332.



pmg/p2

b

Fig. 7. (continued).

COXETER, H. S. M. (1986) Coloured symmetry, in *M. C. Escher: Art and Science* (eds. H. S. M. Coxeter *et al.*), pp. 15–33, North-Holland, Amsterdam.

CROWE, D. (1975) The geometry of African art II, *Historia Mathematica*, **2**, 253–271.

GERDES, P. (1989) Reconstruction and extension of lost symmetries, in *Symmetry 2. Computers Math. Applic. CAMWA* (ed. I. Hargittai), **17**, No. 4–6, pp. 791–813, Pergamon Press, Oxford.

GEREVICH, T. (1938) *Magyarország románkori emlékei* (*Romanesque Art Heritage of Hungary*), Királyi Magyar Egyetemi Nyomda, Budapest (in Hungarian).

GHIRSHMAN, R. (1964) *Persia. From the Origins to Alexander the Great*, Thames and Hudson.

HENSZLMANN, I. (1876) *Magyarország ó-keresztény, román és átmeneti stílü müemlékeinek rövid ismertetése* (*Description of the Architectural Monuments of Hungary in the Old-Christian, Romanesque and Transitional Styles*), Budapest (in Hungarian).

HOOD, S. (1978) *The Arts in Prehistoric Greece*, Penguin, Harmondsworth.

MACGILLAVRY, C. H. (1976) *Symmetry Aspects of M. C. Escher's Periodic Drawings*, International Union of Crystallography, Bohn, Scheltema and Holkema, Utrecht.

Prince MIKASA TAKAHITO (1996) *Development and Modification of the "Holy Symmetrical Design" West and East of the "Silk Road"* (eds. Katachi U Symmetry, T. Ogawa, K. Miura, T. Masunari and D. Nagy), pp. 81–94, Springer, Tokyo.

SHUBNIKOV, A. V. and KOPTSIK, V. A. (1977) *Symmetry in Science and Art*, Plenum Press, New York and London.