Sierpinski Gaskets in Excitable Media

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Abstract. In our previous papers, we have shown by computer simulations that a Sierpinski gasket pattern appears in a Bonhoffer-van der Pol type reaction diffusion system. In this paper, we show another class of regular self-similar structure which is found in four different excitable reaction diffusion systems. This result strongly implies that the existence of the self-similar spatio-temporal evolution is universal in excitable reaction-diffusion media.

1. Introduction

Recently a rich variety of pulse dynamics has been found in nonlinear open systems. Especially, collision and self-replication of pulses has been investigated in reaction diffusion systems. Computer simulations of several reaction-diffusion equations have revealed that a propagating pulse is stable upon collision with another pulse. That is, a pair of counter-propagating pulse undergoes an elastic-like collision (PETOV et al., 1994; OHTA et al., 1997). It is also possible that a pulse pair is deformed during collision but survives again just like a soliton in an integrable non-dissipative system (KOSEK and MAREK, 1995; HAYASE, 1997). Self-replication of pulses has also been found by simulations. In the Gray-Scott model, a motionless pulse splits into two pulses which grow and repeat self-replication until the density of pulses is sufficiently large (PETOV et al., 1994; NISHIURA et al., 1995). A propagating pulse can also self-replicate in which a pulse is emitted as a back-firing (PETOV et al., 1994; IMURA and NAGAYAMA, 1997).

It is important to note that three basic properties of pulses, pair annihilation and preservation upon collision and self-replication can coexist in a small but finite parameter region. In this situation, we have shown in previous papers (HAYASE, 1997; HAYASE and OHTA, 1998) that the interplay among the three properties causes a regular self-similar spatio-temporal evolution of a trajectory of pulses. An extinction of pulses except for the edges occurs every three generations. This is isomorphic to a Sierpinski gasket (SG) generated by cellular automaton.
\begin{equation}
\alpha^{i+1} = \alpha^{i-1} + \alpha^{i+1}, \quad \text{mod} k
\end{equation}

where \( \alpha(i) = 0, 1, \ldots, k - 1 \) defined on a one-dimensional lattice. Actually the pattern in previous papers (Hayase, 1997; Hayase and Ohta, 1998) corresponds to the case \( k = 3 \).

The purpose of this paper is to explore how generic the formation of a regular self-similar pattern is. We will show that the SG is not an exceptional one for a particular set of reaction-diffusion equations. In four different model systems, a self-similar pattern equivalent with Eq. (1) with \( k = 2 \) will be obtained. The results definitely indicates that the existence of the self-similar spatio-temporal evolution is universal in excitable reaction-diffusion media.

In Secs. 2–5, we introduce four different excitable reaction-diffusion systems which SG can be produced. Discussion is given in Sec. 6.

2. BVP Model with a Cubic Nonlinear Term

In our previous papers (Hayase, 1997; Hayase and Ohta, 1998), we have reported that a regular self-similar spatio-temporal pattern like an SG appears in a Bonhoffer-van der Pol (BvP) type reaction-diffusion equations,

\begin{align}
\tau \frac{\partial u}{\partial t} &= D_u \frac{\partial^2 u}{\partial x^2} + f(u) - v, \\
\frac{\partial v}{\partial t} &= D_v \frac{\partial^2 v}{\partial x^2} + u - \gamma v + I,
\end{align}

where positive constants \( D_u \) and \( D_v \) are the diffusion rate of \( u \) and \( v \), respectively. The parameters \( \tau \), \( \gamma \) and \( I \) are positive constants. In this section we shall explore the pulse dynamics of BvP model (2) and (3) with a cubic nonlinearity

\begin{equation}
f(u) = a u (u + 1)(1 - u),
\end{equation}

where \( a \) is a positive constant. The parameters are chosen such that the system Eqs. (2) and (3) with (4) is excitable. Throughout this section, we set \( D_u = 1, D_v = 10, a = 5, \gamma = 0.25 \) and \( I = 0.1 \). We examine the behavior of pulses by changing the values of \( \tau \).

Equations (2)–(4) have been studied in detail theoretically (Rinzel and Keller, 1973; Koga and Kuramoto, 1980; Ito and Ohta, 1992). As shown in Fig. 1, Eqs. (2)–(4) have a stable traveling pulse when \( \tau < \tau_p \), whereas a stable motionless pulse exists when \( \tau > \tau_m \).

In the region \( \tau_p < \tau < \tau_m \), a motionless pulse looses stability and the breathing motion appears. It should be noted that there is an interval \( \tau_p < \tau < \tau_b \) where neither motionless pulse nor traveling pulse exist. This is the very region where self-replication of pulses is observed and hence rich varieties of spatio-temporal patterns appear.

When the value of \( \tau \) is in the middle of the region, a regular self-similar pattern like a Sierpinski gasket appears as in Fig. 2. Note that the SG in Fig. 2 differs from that reported
previously (HAYASE, 1997; HAYASE and OHTA, 1998). Most crucial property is that preservation of pulses does not occur in Fig. 2. All of the pulses undergo pair-annihilation so that extinction of pulses except for the edge pulses occurs every two generations. This SG is equivalent with (1) with $k = 2$.

3. The BvP Model with a Hyperbolic Tangent Nonlinear Term

In the preceding section, we have obtained an SG with $k = 2$ in the excitable system (2)–(4). This is quite contrast to our previous result that an SG with $k = 3$ was obtained in Eqs. (2) and (3) with a hyperbolic tangent nonlinearity,

$$f(u) = \frac{1}{2} \left[ \tanh \frac{u - \alpha}{\delta} + \tanh \frac{\alpha}{\delta} \right] - u$$

(5)

where $\alpha$ and $\delta$ are positive constants. The essential difference is that Eqs. (2) and (3) with (5) is bistable in a sense that a uniform stable solution and a stable limit cycle solution coexists.
In this section, we will show that an SG with \( k = 2 \) can be realized even for the hyperbolic tangent nonlinearity. We have carried out numerical simulations of Eqs. (2) and (3) with (5) in the parameter region where \( \gamma = 0.21 \) and other parameters are chosen to be almost same as of the SG with \( k = 3 \). An SG with \( k = 2 \) is really obtained as shown in Fig. 3.

4. The Gray-Scott Model

In Sec. 2 and Sec. 3, we have shown that the SG with \( k = 2 \) appears in the BvP model with two different nonlinear terms. In this section, it will be shown that the SG emerges the Gray-Scott model given by the following set of equations.

\[
\frac{\partial u}{\partial t} = D_u \frac{\partial^2 u}{\partial x^2} - uv^2 + F(1-u)
\]

(6)

\[
\frac{\partial v}{\partial t} = D_v \frac{\partial^2 v}{\partial x^2} + uv^2 - (F + k)v
\]

(7)

where \( D_u \) and \( D_v \) are the diffusion coefficients. \( F \) and \( k \) are positive constants.

Self-replication of a pulse in the Gray-Scott model has been studied both numerically and analytically (PETOV et al., 1994; NISHIURA and UEYAMA, 1999; PEARSON, 1993). However, the spatio-temporal evolution of the interacting pulses has not been attracted much attention. We carry out simulations of (6) and (7) and obtain an SG with \( k = 2 \) as shown in Fig. 4.

5. The Prague Model

The fourth example of excitable systems where an SG appears is the following two-component reaction-diffusion model

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{1}{\varepsilon} \left[ c - u + (1 - c) v \right] \left( u - \frac{v + b}{a} \right).
\]

(8)

\[
\frac{\partial v}{\partial t} = D \frac{\partial^2 v}{\partial x^2} + u - v
\]

(9)

where \( D \) is diffusion coefficient and \( a, b, c \) and \( \varepsilon \) are positive constants.

KASTANEK et al. (1995) have introduced (8) and (9) and studied numerically to simulate splitting of a reduction wave in Belousov-Zhabotinski reaction. We call this model (8) and (9) the Prague Model. They have found that self-replication of pulses occurs for \( a = 0.99, b = 0.01, c = 0.2, D = 1 \) and \( \varepsilon = 0.01 \). Here, we have investigated the behavior of the Prague model by changing the value of \( D \) and \( \varepsilon \). An SG with \( k = 2 \) appears for \( D = 1.2 \) and \( \varepsilon = 0.01 \) as shown in Fig. 5.
Fig. 3. SG for Eqs. (2), (3) and (5). The parameters are $\alpha = 0.105$, $\gamma = 0.21$, $\delta = 0.05$, $\tau = 0.4$, $D_v = 10.5$ and $D_u = 1$. The lines indicate the contour line of $u = 0.2$.

Fig. 4. SG for the Gray-Scott model. The parameters are $F = 0.0253$, $k = 0.0525$, $D_u = 1.15 \times 10^{-5}$ and $D_v = 1.0 \times 10^{-5}$. The lines indicate the contour line of $u = 0.5$. 
6. Discussion

In this paper, we have shown by numerical simulations, that Sierpinski gasket pattern with $k = 2$ can be produced by four different excitable reaction-diffusion systems, (i) the BvP model with a cubic nonlinear term, (ii) the BvP model with a hyperbolic nonlinear term, (iii) the Gray-Scott model and (iv) the Prague model. We expect from these results that the SG is very common, characterizing pulse dynamics in excitable media.

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REFERENCES