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Folding Tetrahedra and Four-Dimensional Origamis

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Abstract. By pure analogy with a usual three-dimensional origami, we can fold a regular octahedral material along flat faces in 4-space. The octahedron comprises four congruent tetrahedra. We will show a procedure to fold a tetrahedron along bisectors of the dihedral angles. This procedure demonstrates that each two of four surfaces of the given tetrahedron coincide with each other and that the point of intersection of those bisectors is the center of the incircle. We will prove that there is a kind of a folded tetrahedron whose flaps swing freely. Consistently joining such folded tetrahedra which construct the regular octahedron, we obtain a four-dimensional bird-base. As its cross-section in 3-space, we can see a traditional bird base. Concerning the three-incircle theorem on a triangle found by HUSIMI and HUSIMI (1979), we will prove the similar theorem for one of four congruent tetrahedra which consist of an octahedron having the 4-fold symmetry.

1. Introduction

For origami artists, a square is the origin of every form. It has two kinds of mirror-symmetry along its diagonals and along its vertical bisects of the edges. A base is a term used for a shape made from the original square that gives rise to a variety of models. A bird-base is the source of the traditional crane origami and its folding pattern comprises four rectangular-equilateral triangles.

During the process of getting the bird base from a preliminary base, two sides of the flap touch and the flap lies flat while the flap has stretched. As pointed out by HUSIMI and HUSIMI (1979), the fact that the flap lies flat and swings freely like a hinge is closely related to the incenter theorem on a triangle.

They tried to make an origami crane by using a quadrilateral which is shaped like a kite with two lower sides longer than two higher sides. First they proved the three-incenter theorem on a triangle which gives us how to find the center of the crane made from this quadrilateral. Thus they obtained an origami crane from a kite-shaped quadrilateral and developed a flying crane from it (HUSIMI and HUSIMI,1984). This theory about a variety of origami cranes has been extensively developed by JUSTIN (1994) and KAWASAKI (1995a, 1998).

As a four-dimensional origami we use a regular octahedron as a basic material as in Fig. 1(a) and fold it along flat faces in 4-space determined by the *xyzu*-coordinates (KAWASAKI, 1995b). MIYAZAKI (1997) made a 4-dimensional “Noshi” from a regular octahedron, where a “Noshi” is the traditional Japanese model of the folding. Our problem is to make a 4-dimensional bird-base by means of the incenter theorem. In Sec. 2 we will explain how to fold a tetrahedron and prove that there is a folded model whose flaps swing freely. In Sec. 3 we will show a procedure for getting the four-dimensional bird base. In Sec. 4, we will prove the three-incenter theorem for one of four congruent tetrahedra that comprise an octahedron having the 4-fold symmetry about its diagonal AC as in Fig. 1(b).

2. Folding Tetrahedron around the Incenter

By pure analogy with a three-dimensional origami we use a regular octahedron as the basic material and fold it along flat faces in 4-space. A tetrahedron in 4-space corresponds to a triangle in 3-space. We will show how to fold a tetrahedron by using the incenter theorem on a tetrahedron. As a typical example, let us take one of four congruent tetrahedra which comprise a regular octahedron. Figure 2 shows this tetrahedron ABCD, where I is the incenter, M the midpoint of BD, and Q, R, S and T the tangency points at which the sphere inscribes in ABCD touches its four faces. Each time the tetrahedron is folded it is divided into two or more small tetrahedra. The *u*-coordinate is supposed to be in a vertical direction because it is perpendicular to the *xyz*-space. Then new small tetrahedra lie in different heights along the *u*-direction and are represented by a tree structure with nodes (UCHIDA and ITOH, 1991). The depth of the tree means the step of the folding procedure. The procedure for folding the tetrahedron ABCD is as follows (see Fig. 2):

1) Fold the top half ACDM to the bottom ABCM.
2) Inside reverse-fold ABMI and BCMI while crimp-folding BTMI. (for these two folds see ENGEL (1989))
3) The completed fold.

First, the vertex D is superposed to B in 4-space and the tangency point R to Q. Then the midpoint M is superposed to E and the tangency points S and T to Q. Note that B, Q and E lie on the plane ABC. We can see that in Fig. 2(c) the four flat faces of the original tetrahedron lie on the plane ABC. In Fig. 3, heights of the small tetrahedra are represented

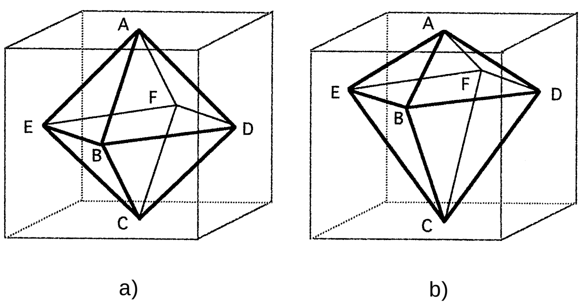

Fig. 1. Octahedra, a) a regular octahedron and b) a kite-shaped octahedron.

as in fractions related to a scale of the arbitrary small unit, whose information shows the stacking order of the small tetrahedra along the *u*-direction.

There is an another procedure for folding ABCD. Let us make a hollow model of a tetrahedron from a sheet of paper, and then squash and flatten it. Figure 3 shows that there is a fixed point which coincides with the tangency point Q and that there are two kinds of flatten models. One model shown in Fig. 3(a) has the same stacking order as the completed folding as in Fig. 2. In Fig. 3, we give triangles whose heights are as in Fig. 2. We can use those flatten models of a tetrahedron to study how flaps move. Thus the model shown in Fig. 3(a) has flaps which swing freely around the point Q as a pin. However another model shown in Fig. 3(b) has not such flaps because the flaps are fasten into a pocket. To see this fact in detail let us look into three intersections S_1, S_2 and S_3 between the folded tetrahedron and $x = 0$ plane as in Fig. 3. Those intersections are looked similar to ones that we see when

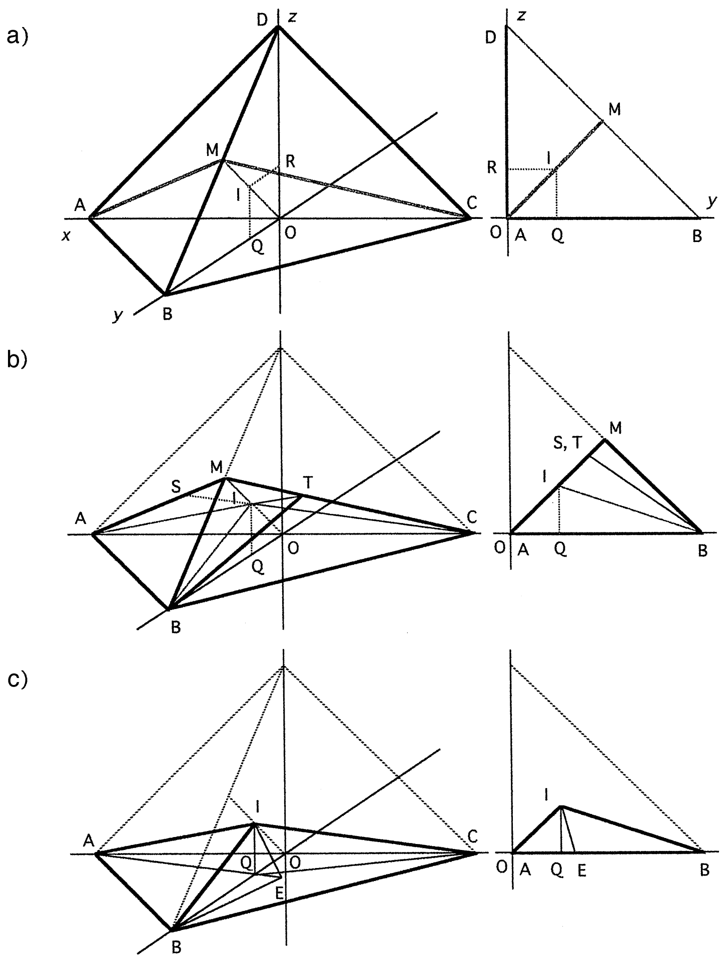

Fig. 2. Procedure of the folding tetrahdron with its *yz*-plane view.

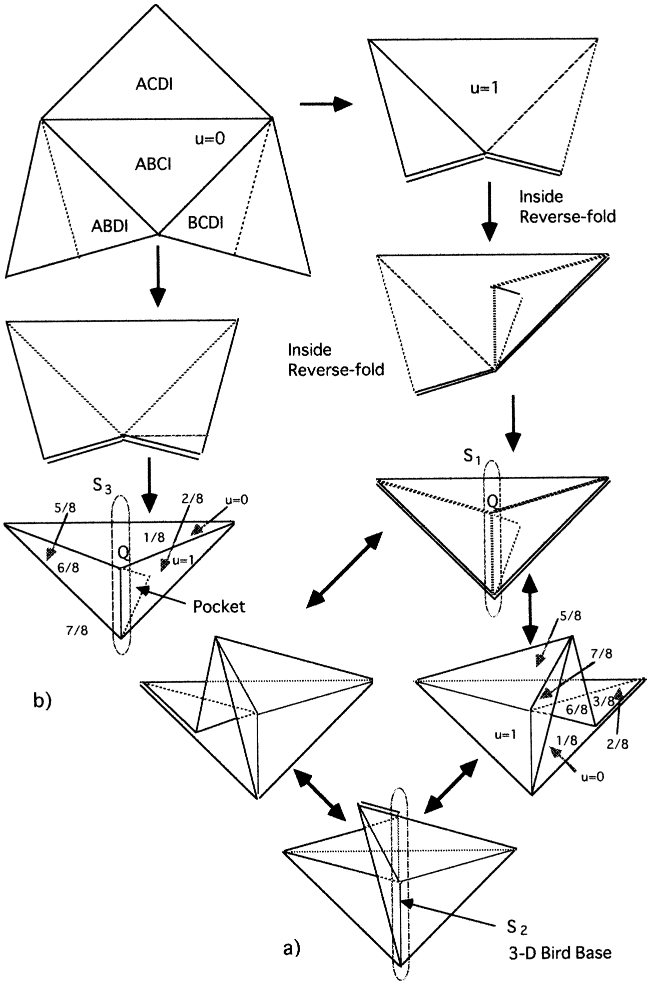

Fig. 3. Two flatten models of a folded tetrahdron, a) models with swinging flaps and b) models with fastened flaps; S_1, S_2 and S_3 are the intersections of those models with the $x = 0$ plane.

the triangle BOD in Fig. 2 is folded using the incenter theorem. Because the incenter of the triangle BOD is different from the incenter of the tetrahedron ABCD, the flaps are lower than those of the folded triangle.

3. Four-Dimensional Bird Base

We have shown by the above four-dimensional origami that the incenter theorem on a tetrahedron is demonstrated by the procedure of folding tetrahedron and that there is a kind of folded tetrahedra whose flaps swing freely. By consistently joining such the folded tetrahedron which comprise an octahedron we will obtain a four-dimensional bird-base.

Let us fold a regular octahedron as in Fig. 1(a). The procedure for obtaining the bird base is the following:

1) Fold the back half AECDF to the front AECDB.

2) First fold ACDM to ABCM and then ACEN to ABCN, where M and N are the midpoints of BD and BE.

3) This is a four-dimensional preliminary fold. Inside reverse-fold ABMI and BCMI while crimp-folding BTMI, where I is the incenter of ABCD. Repeat this folding on the lower half ABCN at the same time.

4) The completed bird-base.

Suppose that at each step we look into the intersection between a folded octahedron and the $x = 0$ plane. Then we can see that the intersection looks like one when we fold a square. On a four-dimensional preliminary base the intersection is the same as a three-dimensional preliminary base. Folding the bird-base and stretching its flaps in 4-space we will see a four-dimensional flapping bird and an origami crane. From various three-dimensional bases (ENGEL, 1989) we will obtain their four-dimensional bases according to the similar way.

4. The Three-Incenter Theorem on a Tetrahedron

To make an origami crane from a quadrilateral whose shape is a kite as in Fig. 4 we start from finding out the center of the crane using the three-incenter theorem (HUSIMI and HUSIMI, 1979).

Theorem 1 (The three-incenter theorem on a triangle) *In △ABC, let L be the incenter of △ABC and O the tangency point at which the incircle touches AB. Divide △ABC into two △ABO and △CBO by BO whose incenters are denoted by M and N respectively. Then MN is perpendicular to OB.*

Let P be the intersection of BO and MN. The key for the proof is that the tangency point O satisfies the relation AB + OC = BC + OA and then OP = (OA + OB – AB)/2 = (OB + OC – BC)/2. JUSTINE (1994) called the point O the center of the crane (the point of Loiseau in French). Let us consider a quadrilateral with inscribed circle (qwic) called by HUSIMI and HUSIMI (1984). He showed that for a qwic ABCD, the center of the crane is the intersection of branches of two hyperbolas given by the relation AB + CD = BC + AD. This theorem gives us how to determine the center of the crane. When bringing MA, MB, MO, NC, NB and NO together by mountain-folds, the triangle MON swings freely with MN as a hinge as in Fig. 4(b).

As a four-dimensional origami let us study an octahedron as in Fig. 1(b) that has rotational symmetry of order four and whose intersections ABCE and ADCF are the kite-shaped quadrilateral. For a quarter of this octahedron, ABCD in Fig. 5, we find out the following theorem.

Theorem 2 (The three-incenter theorem on a tetrahedron) *On the tetrahedron ABCD, let L be the incenter of ABCD and G the tangency point at which the inscribed sphere touches △ABC. Let H be the intersection between BG and AC. Divide the tetrahedron*

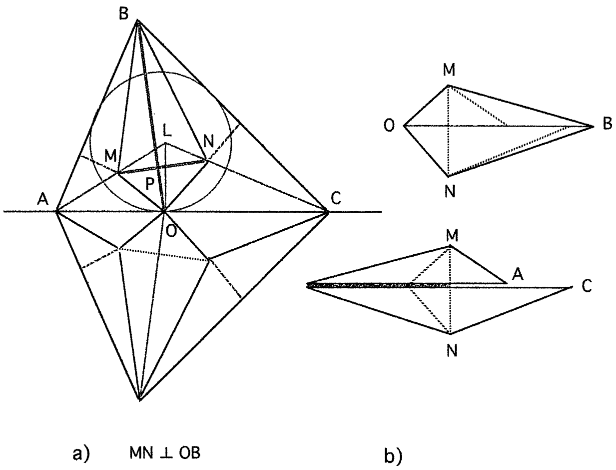

Fig. 4. The bird-base derived from a kite-like quadrilateral and its folding pattern.

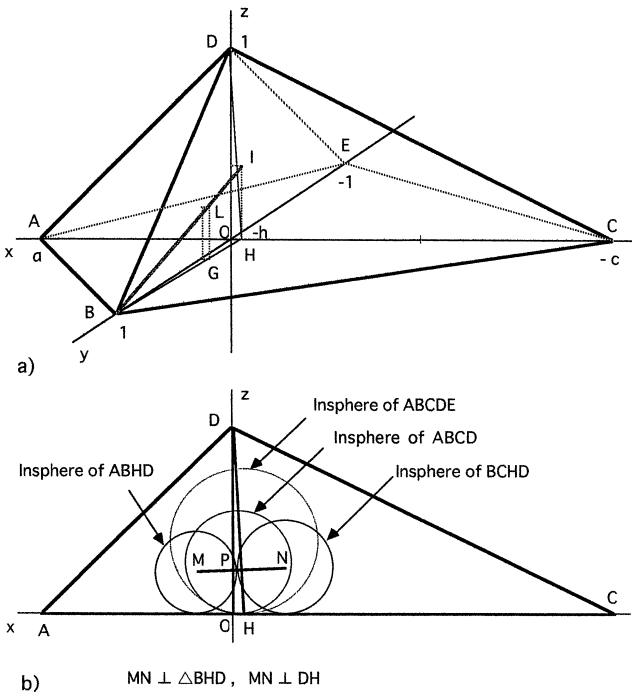

Fig. 5. The three-incenter theorem about a tetrahedron, a) the point of the crane H and b) the *xz*-plane view of the configuration of three inspheres.

ABCD into two small tetrahedrons ABHD and BCHD whose incenters are denoted by M and N respectively. Then MN is perpendicular to △BHD.

Let us assume that OA = a is smaller than OC = c as in Fig. 5. Then the incenter L has the coordinates (s, r, r), where the inradius $r = r(a, c)$ and $s = s(a, c)$ are

$$r(a,c) = \frac{a+c}{2(a+c)+\sqrt{1+2a^2}+\sqrt{1+2c^2}}, \quad s(a,c) = \frac{a\sqrt{1+2c^2}-c\sqrt{1+2a^2}}{2(a+c)+\sqrt{1+2a^2}+\sqrt{1+2c^2}}.$$

As H has the x-coordinate $h = (r - 1)/s$, M has the coordinates (s_1, r_1, r_1) and N ($-s_2$, r_2, r_2), where $r_1 = r(a, h)$, $s_1 = s(a, h)$, $r_2 = s(b, -h)$, $s_2 = s(b, -h)$. Let P be the point at which the inscribed sphere of ABHD touches △BHD, and P′ the point at which the inscribed sphere of BCHD touches △BHD. Using those coordinates we can show that P coincides with P′. Thus the theorem is regarded as the four-dimensional version of Theorem 1.

Now we will show another way to find out the point H. Let I be the incenter of the pyramid ABCDE as in Fig. 5(a) that comprises the tetrahedron ABCD and its reflection about the y = 0 plane. The incenter I is the point of the intersection of the bisector planes of the dihedral angles of ABCD. Two of those bisector planes △IBC and △IDB and the y = 0 plane determine the incenter I. The incenter I is the point at which BL intersects the y= 0 plane. Thus H is its projection in the z = 0 plane. Figure 5(b) shows the xz-plane view of the configuration of the three inspheres in the ABCD. Note that the projection of MN to the y = 0 plane is perpendicular to DH.

Let us make the hollow tetrahedron ABCD and then squash and flatten it. Using this flatten model we will see that the incenter M of ABHD coincides with the N of BCHD.

Using Theorem 2 we can determine the point of the crane and then we will obtain the 4-dimensional bird base by folding the kite-shaped octahedron.

5. Conclusions

We have folded tetrahedra in 4-space and shown how to fold the four-dimensional bird-base from an octahedron. We have obtained the following results.

(1) When a tetrahedron is folded by using the incenter theorem, its four faces lie on one of those face.

(2) The flatten model of tetrahedron has the same stacking order of small parts as the folded tetrahedron has.

(3) There is one kind of the folded tetrahedron whose flaps swing freely.

(4) For one of the four congruent tetrahedra consisting of a kite-like octahedron, the three-incenter theorem is obtained and demonstrated by using a flatten model of the tetrahedron.

REFERENCES

ENGEL, P (1989) *Folding the Universe: Origami from Angelfish to Zen*, Vintage Books, New York, pp. 76–85.
HUSIMI, K. and HUSIMI, M. (1979, 1984 2nd Ed.) *Origami no Kikagaku* (*Geometry of Origami*), Nippon

Hyoronsya, Tokyo, pp. 107–126 (in Japanese).
JUSTIN, J. (1994) Mathematical remarks about origami bases, *Symmetry: Culture and Science*, **5**, 153–165.
KAWASAKI, T. (1995a) Extension of systematic method of constructing flat origami cell decomposition and its application, *Rept. Sasebo Coll. Tec.*, **32**, 29–58 (in Japanese).
KAWASAKI, T. (1995b) Folding space by using freely flat origami method, *Quarterly Oru* (*Folding*), No.8, Sojusya, Tokyo, pp. 99–101 (in Japanese).
KAWASAKI, T. (1998) *Bara to Origami to Sugaku to* (*Rose, Origami and Mathematics*), Morikita, Tokyo, pp. 138–159 (in Japanese).
MIYAZAKI, K. (1997) Four-dimensional origami, in *Proceedings for the Second International Meeting of Origami Science and Scientific Origami* (ed. K. Miura *et al*.), Seian University of Art and Design, Otsu, pp. 51–61.
UCHIDA, T. and ITOH, H. (1991) Knowledge representation of origami and its implementation, *Trans. IPS Japan*, **32**, 1566–1573 (in Japanese).