Are Borromean Links So Rare?

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Abstract. After describing classical Borromean links and their properties, Borromean property and Brunnean property are extended to *n*-component links ($n > 3$). Different known infinite classes of three-component Borromean links are described, as well as two new infinite classes of "prismatic" Borromean *n*-component links.

"No two elements interlock, but all three do interlock". A three-component link with that property is called "Borromean" after the Borromeos, an Italian family from the Renaissance that used them as their family crest symbolizing the value of collaboration and unity. B. Lindström and H. O. Zetterström (LINDSTRÖM and ZETTERSTRÖM, 1991), proved that "Borromean circles are impossible": three flat circles cannot construct them, but by triangles they can. The Australian sculptor J. Robinson assembled three flat hollow triangles to form a structure (called *Intuition*), topologically equivalent to Borromean rings. Their cardboard model collapses under its own weight, to form a planar pattern. P. Cromwel recognized Borromean triangles in a picture-stone from Gotland (CROMWEL, 1995). This and other symmetrical combinations of three and four hollow triangles were considered by H. S. M. Coxeter (COXETER, 1994). In geometry, Borromean rings appear as the regular octahedron {3,4} (JABLAN, 1998), in Venn diagrams (RUSKEY, 1999), in DNA (SEEMAN, 1999), and in other various areas (CROMWEL *et al.*, 1998) (Fig. 1).

In the knot theory Borromean rings are the foremost examples having with two remarkable properties: three mutually disjoint simple closed curves form a link, yet no two curves are linked, and if any one curve is cut, the other two are free to separate. In the case of 3-component links those two properties are inseparable: one follows from the other. In the case of *n*-component links ($n > 3$), *n*-Borromean links could be defined as *n*-component nontrivial links such that any two components form a trivial link. Among them, those with at least one nontrivial sublink, for which we will keep the name "Borromean links", will be distinguished from the Brunnian links in which every sublink is trivial (LIANG and MISLOW, 1994).

It seams surprising that besides the Borromean rings, represented by the link 6_2^3 in Rolfsen's notation, no other link with the properties mentioned above can be found in link tables (ROLFSEN, 1990; ADAMS, 1994). The reason for this is very simple: all existing knot tables contain just the links with at the most 9 crossings. In fact, an infinite number of Borromean or Brunnian links exist, and they can be derived as infinite series.

The first such series of 3-component links, beginning with the Borromean rings, was discovered by P. G. Tait (TAIT, 1876–77). Their geometrical source is easy to recognize: the limit case for $n = 1$ yields the regular octahedron {3,4}, while the series of ($3n$)-gonal antiprisms are obtained for $n > 1$. Alternating their corresponding projections (well known in geometry as Schlegel diagrams) provides the series of achiral 3-Borromean links (Fig. 2).

If it is not necessary that every two components in a projection do intersect, an infinite number of "fractal" Borromean links derive from each n-Borromean link in a very simple way. Indeed, it is enough to surround in a projection an even number of the appropriately chosen crossing points of any two components by circles (Fig. 3). Therefore, our considerations will be restricted to n-Borromean links without nonintersecting components in a projection.

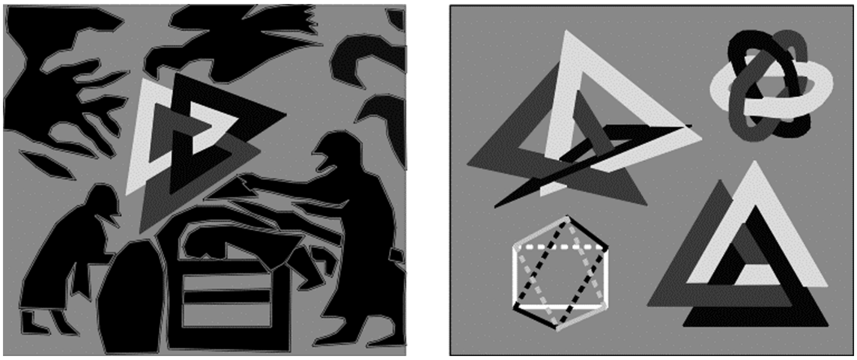

Fig. 1. Detail from a picture-stone on Gotland, hollow triangles, Borromean rings and regular octahedron.

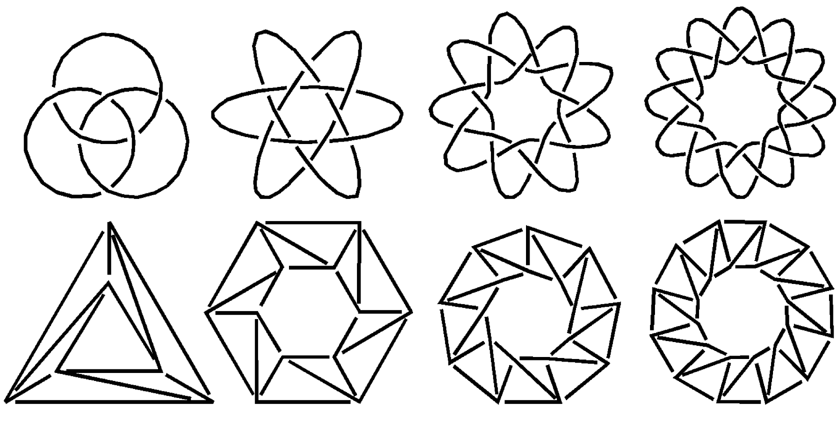

Fig. 2. Tait series.

The next infinite series of 3-Borromean links, beginning again with the Borromean rings, follow from a circular 2-component trivial links by introducing the third component: a circle intersecting the projection in opposite points (Fig. 4). In a similar way, from the family of 2-component trivial links we derive the other infinite series of 3-component Borromean links (Fig. 5). From such links with a self-crossing projection of a component, new infinite series of Borromean likns with twists are obtained. In a self-crossing point of the oriented component projection an n-twist is introduced, its orientation being used only for choosing the appropriate position of the twist (Fig. 6).

Note that the first series of Borromean links with twists (Fig. 4) could be also derived from Borromean rings by introducing identical twists in the crossing-points of two different components. Therefore, we could first get different infinite series of n-Borromean

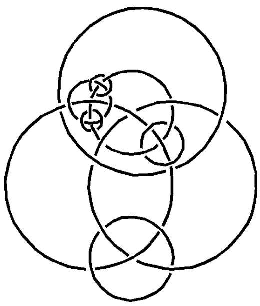

Fig. 3. Fractal Borromean links.

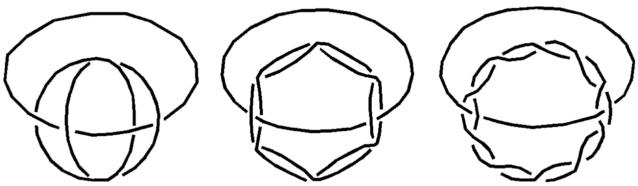

Fig. 4. Borromean links derived from circular links.

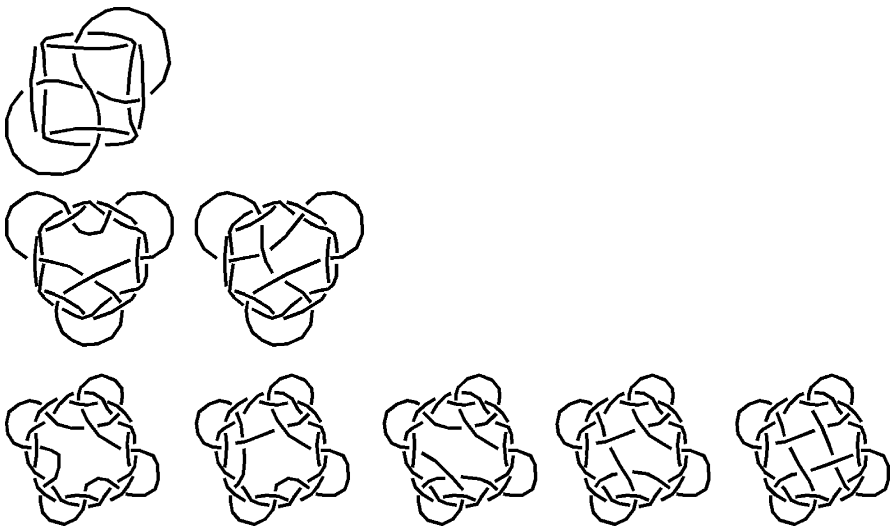

Fig. 5. Another series of Borromean links derived from the same source.

links without twists, and then introduce twists trying to preserve the Borromean property.

Tessellations of $(2n + 1)$-gonal prism, where in every ring of the projection we draw "left" or "right" diagonals (Fig. 7), yield the next infinite series. The notation of the tessellations and links can be done using the symbols $(2n + 1, \underline{k})$, where all decompositions

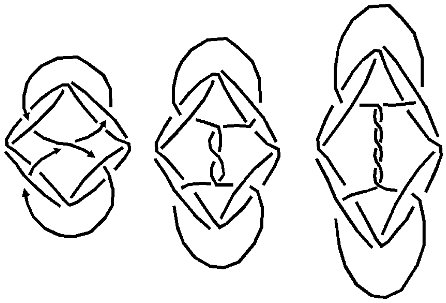

Fig. 6. Introduction of a twist.

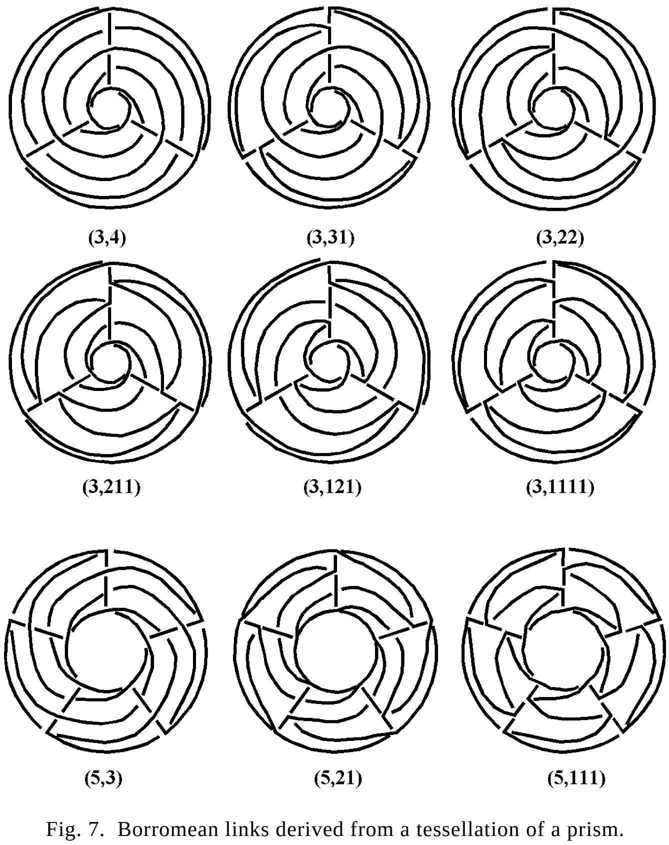

Fig. 7. Borromean links derived from a tessellation of a prism.

of the number $k = (2n + 1)l$-2, are denoted by $\underline{k}$, such that every decomposition is identified with its obverse. From each such tessellation with k rings we obtain $2^{k-2} + 2^{[k/2]-1}$ different $(2n + 1)$-Borromean links without the Brunnian property. In those links, all components are equivalent, i.e. there is an isotopy of 3-space that carries the link to itself and any given component onto any other. Then the digons could be introduced again, in the same way as before. The same method, using "centered" rectangular tessellations, provides another series of $(2n + 1)$-Borromean links (Fig. 8).

Next, we could try to construct Borromean links with an even number of components and without the Brunnian property. C. Liang and K. Mislow (LIANG and MISLOW, 1994) proposed two methods for the construction of n-Borromean links with at least one nontrivial sublink, both resulting in n-Borromean links with some nonintersecting component projections ($n > 3$). In the first method, involving duplication of one or more rings, the duplicate rings are interchangeable by continuous deformation. For example, by duplicating one ring in Borromean rings, we obtain 4-Borromean link, and continuing in the same

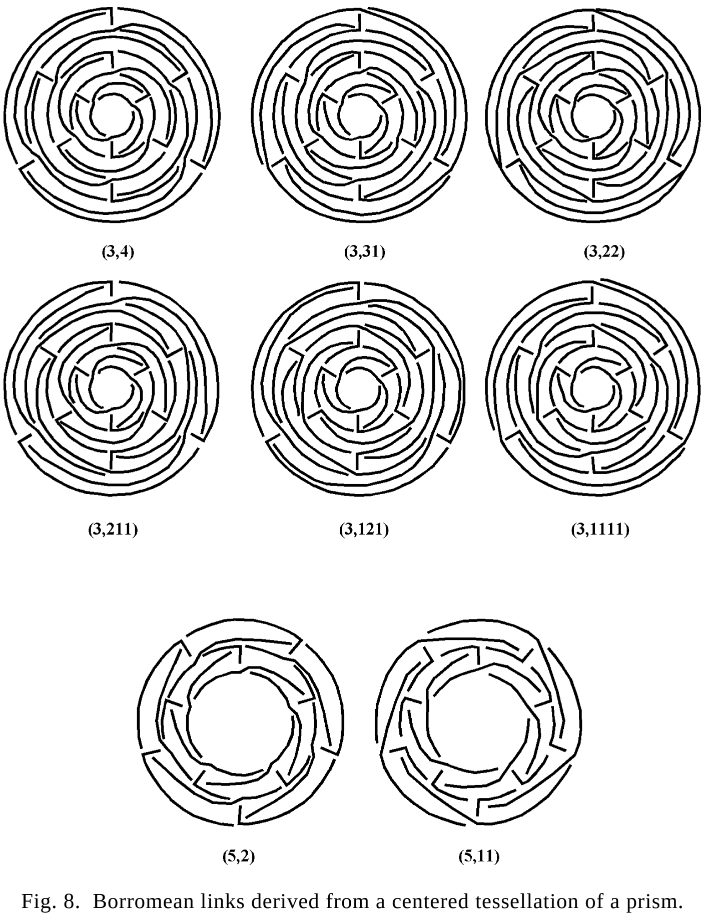

Fig. 8. Borromean links derived from a centered tessellation of a prism.

manner, n-Borromean links (n = 5, 6, 7, ...) (Fig. 9). Different links of that infinite series follow from other choices of rings that will be duplicated. Another method is similar to the one for producing "fractal" Borromean rings: in the trivial link, two crossing points in a projection are surrounded by nonintersecting circles (Fig. 10). Continuing in this way, Borromean links with an even number of components are obtained. Finaly, only one open question remains: are they exist ($2n$)-Borromean links in which every pair of projections of components has a crossing in all projections of the link, and moreover, where all components are equivalent.

The n-component links ($n > 3$) without nontrivial sublinks were described by H. Brunn (BRUNN, 1892). Here they are presented by a series of illustrations with some artistic qualities (SCHAREIN, 1998) (Fig. 11).

At the end, let us consider different plane arrangements of circles, where every intersection or touching point is common for exactly two circles (Fig. 12). Any such arrangement is four-valent, and by alternating it can be transformed into a projection of some knot or link. Different arrangements of circles may result in the isomorphic knot or link projections. For example, one arrangement of three circles, and the other of four circles may result in the same projection of two-component link 5_1^2. Therefore, for every circle arrangement we need to consider the number of circles from which it consists and the knot or link projection obtained.

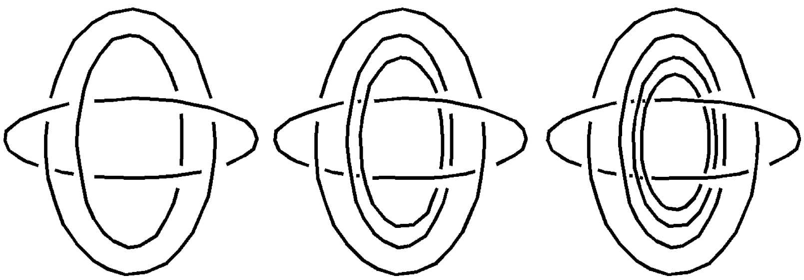

Fig. 9. Borromean links derived by ring duplication.

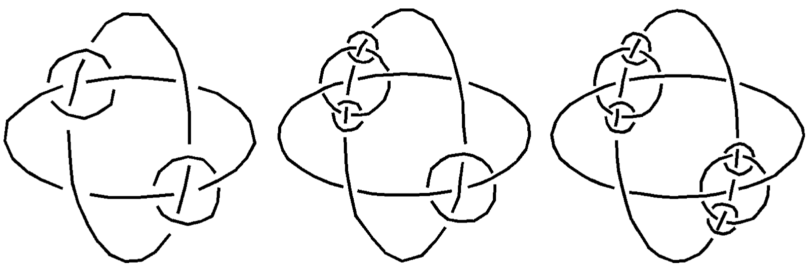

Fig.10. The other method of construction.

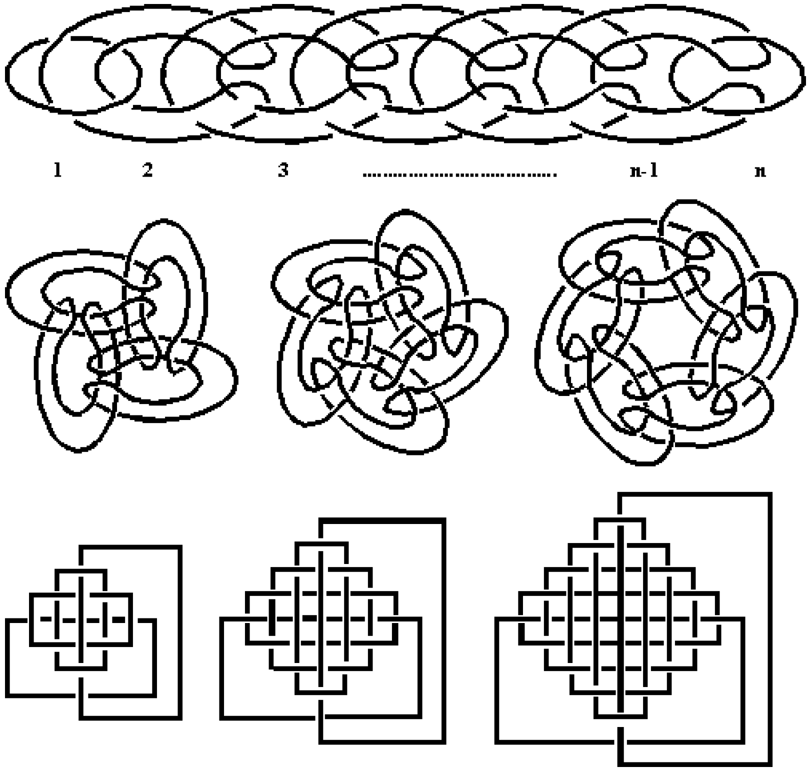

Fig.11. Brunnean links.

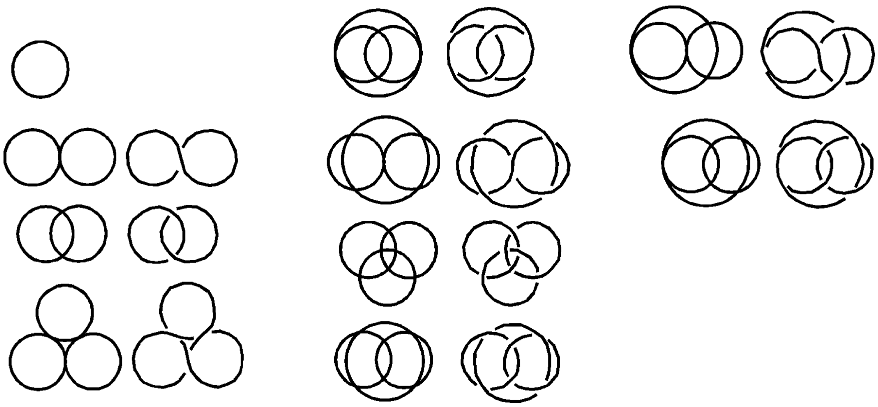

Fig.12. Circle arrangements, knots and links derived from them.

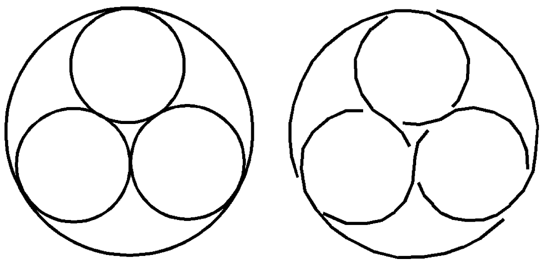

Fig.13. Four-circle arrangement producing Borromean rings.

Some interesting possibilities, similar to certain “wasan”-patterns (NAGY, 1995) arrise in the case of inscribed cicles. For example, except the “classical” Borromean rings formed by three circles (known also as one of old Japanese family crests), Borromean rings can be derived from the following four-circle arrangement (Fig. 13).

In the same way, from the *Mon* pattern (family crest) “Nine Stars” (HUSIMI, 1996) we may obtain the 16-crossing knot corresponding to the 8-antiprismatic basic polyhedron.

In the other paper by Prof. K. Husimi (HUSIMI, 1994) we can find another interesting application of Borromean links: let us put on the tetrahedron surface three rubber rings all through the centers of the edges of the tetrahedron, alternating and forming Borromean rings. The string pattern printed that way on the plane by rolling tetrahedron will be *Kagome*—woven bamboo pattern. The same traditional woven patterns are used by P. Gerdes for modelling the structure of Fullerenes (GERDES, 1998; JABLAN, 1999).

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