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Spectral Analysis of Form Based on Fourier Descriptors

Yoshinori Uesaka

Faculty of Science and Engineering, Science University of Tokyo

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When we memorize a form in our mind and remember it after a few minutes, we can reproduce only a rough sketch of the form without the details contained in the original pattern. So, generally speaking, the information involved in a form may be said to consist of two part : Dominant information and Details. In the present paper, a method how to extract such a dominant information from a given form is proposed. It is also shown that the dominant information of a pattern can be characterized by the low frequency part of the spectrum of the Fourier descriptors. The method to get a rough pattern consists of three kinds of mappings : P-transformation, Fourier transformation and Low-pass filter.

INTRODUCTION

First of all, look at the patterns shown in Fig.1. If you are familiar with a map of Japan, you can easily recognize the pattern (a) in Fig.1 as Hokkaido which is located in the north part of Japan. On the other hand, it might be rather difficult for some of you to recognize the pattern (b) in Fig.1. Because it is one of chinese characters, i.e. Kanji, which is drawn in a cursive hand or running style.

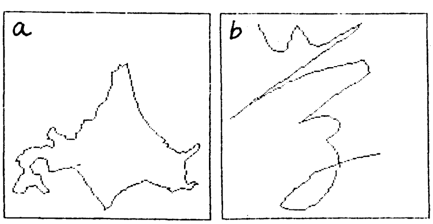

Fig.1 Examples of patterns.

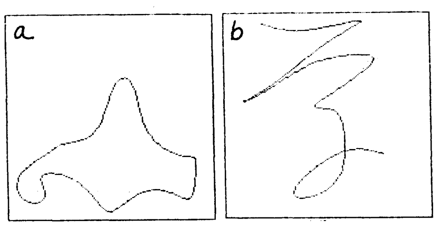

Fig.2 Rough sketches of Fig.1.

Next, watch the second group of patterns shown in Fig.2. Whether you can recognize the patterns in Fig.1 or not, you can see without difficulty that the patterns in Fig.2 are rough scketches of ones in Fig.1, respectively. If you are asked to

Yamazaki 2641, Noda City, Chiba 278, JAPAN.

memorize the patterns in Fig.1 in your mind and to remember them after a few minutes, then you will reproduce only the rough scketches, for instance, like ones in Fig.2 without the details contained in the original patterns.

So, generally speaking, the information contained in a pattern may be said to consist of two parts :

$$\text{Pattern} = \text{Dominant Information} + \text{Details}. \qquad (1)$$

Only the first part, i.e. the dominant part is stored in our brain and can be used to reproduce a rough pattern. The second part, i.e. the details which are very difficult to memorize in our brain. Thus, we can ask what sort of information can characterize the rough pattern like ones in Fig.2.

The main aim of the present paper is to show a sort of model of the rough pattern and to give a method how to get such a rough pattern from a given pattern. As shown later, the spectral analysis based on the Fourier transformation plays an important role to build up a rough pattern model.

1. P-EXPRESSION OF CURVE

In order to construct the rough pattern model, we should prepare several kinds of transformations from pattern to pattern. The first transformation to be introduced is the P-transformation which maps a curve to its P-expression.

To explain the P-transformation, several notations used in the present paper and the notion of P-expression of a curve will be given in this chapter.

A planner curve C may generally be expressed by the set of points (x(s), y(s)) :

$$C = \{ (x(s),\ y(s)) \mid 0 \le s \le 1 \}, \qquad (2)$$

where x(s) and y(s) are the functions, having x-y coordinates as their values, respectively, of the length s of the curve from a point on C (or the end point in the case of an open curve). We also identify the x-axis and the y-axis with the real axis and the imaginay axis, respectively. Thus the plane considered is identified with the Gaussian one. Let i denotes the imaginary unit $\sqrt{-1}$. Hence a complex number

$$z(s) = x(s)+iy(s) \qquad (3)$$

will be identified with a point (x(s),y(s)). The curve C can be therefore identified with the complex-valued function z.

Let us consider an open curve C (Fig.3) which is piecewise smooth. It consists of n smooth subcurves $C_1, \cdots, C_n$. Taking a pass which starts at an end point of C_1 and arrives at an end point of C_n, by s we denotes the length of the curve from the starting point to a current point. Let s_j denote the length from the starting point of C to the end point of C_j. By a_j we mean the angle spanned by the two tangential lines at the end point of C_j, where a_0 is the angle between the tangential line of C_1 and the horizontal coordinate. It is called the **bending angle** and is to be measured counter-clockwise.

Since we assumed that each subcurve C_j is smooth, we have the curvature $\kappa_j(s)$ on the point s on C_j. We assume that $\kappa_j(s)$

is positive or negative if the center of the curvature exists in the lefthand or righthand of the subcurve C_j when we proceed from

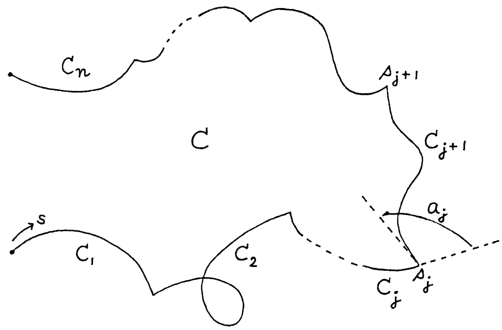

Fig.3 An open curve being piecewise smooth

the starting point to the end point of C_j, respectively. Then, the total curvature function θ is defined recursively as, for s in $[s_0, s_1]$,

$$\theta(s) = a_0 + \int_{s_0}^{s} \kappa_1(s)ds, \tag{4}$$

and for s in $(s_{j-1}, s_j]$,

$$\theta(s) = \theta(s_{j-1}) + a_{j-1} + \int_{s_{j-1}}^{s} \kappa_j(s)ds \quad (j=2,\cdots,n). \tag{5}$$

Fig.4 shows an example of a total curvature function. As seen in this example, the total curvature function is globally decreasing because the curve (a) in Fig.4 is globally clockwise.

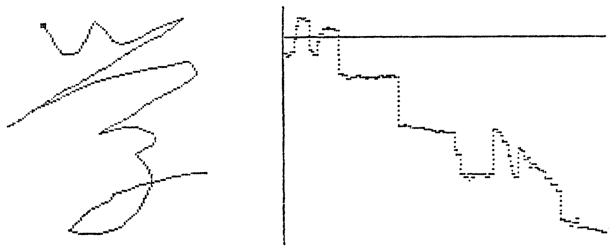

Fig.4 An example of total curvature function

Using the total curvature function, we shall now define the P-expression w of the curve C as

$$w(s) = \exp[i\theta(s)], \tag{6}$$

where i denotes the imaginary unit $\sqrt{-1}$. Now we have got the P-transformation P which maps a curve z to its P-expression :

$$P(z) = w. \tag{7}$$

When we regard w in (6) as the signal of the curve in the sense of communication engineering, then θ may correspond to the phase angle of the signal. That's why we use the letter P in the name of this transformation.

As easily seen, the curve C or z is given by the

P-expression w as follows :

$$z(s) = z(0)+\int_0^s w(t)dt. \tag{8}$$

Thus, given the P-expression w, the curve z of which P-expression is w is uniqely determined up to translation. So, under fixed z(0) we have got the inverse P-transformation P^{-1} :

$$P^{-1}(w) = z. \tag{9}$$

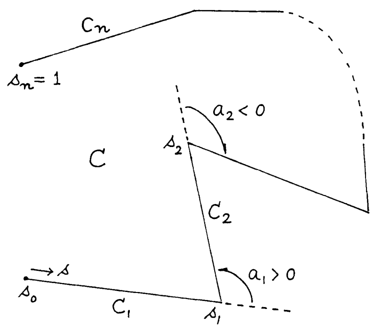

Fig.5 A curve consisting of only line segments

As an example we shall consider the special case of a curve which consists of only line segments (Fig.5). In this case each curvature $\kappa_j(s)$ is constantly equal to 0. Hence the total curvature function becomes more simple as

$$\left.\begin{array}{ll} \theta(s) = a_0 & \text{for } s \text{ in } [s_0, s_1], \\ \theta(s) = \theta(s_1)+a_1 & \text{for } s \text{ in } (s_1, s_2], \\ \cdots\ \cdots\ \cdots & \\ \theta(s) = \theta(s_{n-1})+a_{n-1} & \text{for } s \text{ in } (s_{n-1}, s_n], \end{array}\right\} \tag{10}$$

where $s_0 = 0$ and $s_n = 1$, the total length of a curve. When we write the sum of bending angles $a_0, \cdots, a_j$ as θ_j, the total curvature function and the P-expression are simply written, respectively, as

$$\left.\begin{array}{ll} \theta(s) = \theta_0 & \text{for } s \text{ in } [s_0, s_1], \\ \theta(s) = \theta_1 & \text{for } s \text{ in } (s_1, s_2], \\ \cdots\ \cdots & \\ \theta(s) = \theta_{n-1} & \text{for } s \text{ in } (s_{n-1}, s_n], \end{array}\right\} \tag{11}$$

and

$$\left.\begin{array}{ll} w(s) = \exp[i\theta_0] & \text{for } s \text{ in } [s_0, s_1], \\ w(s) = \exp[i\theta_1] & \text{for } s \text{ in } (s_1, s_2], \\ \cdots\ \cdots\ \cdots & \\ w(s) = \exp[i\theta_{n-1}] & \text{for } s \text{ in } (s_{n-1}, s_n]. \end{array}\right\} \tag{12}$$

Noting that w(s) is constant while s is in $(s_{j-1}, s_j]$, we can easily compute the curve z from the P-expression w :

$$\left.\begin{array}{ll} z(s) = z(0)+(s-s_0)\exp[i\theta_0] & \text{for } s \text{ in } (s_0, s_1], \\ z(s) = z(s_1)+(s-s_1)\exp[i\theta_1] & \text{for } s \text{ in } (s_1, s_2], \\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots & \\ z(s) = z(s_{n-1})+(s-s_{n-1})\exp[i\theta_{n-1}] & \text{for } s \text{ in } (s_{n-1}, s_n]. \end{array}\right\} \tag{13}$$

The reason why we use the P-expression instead of the curve itself is due to the vision psychology rather than a mathematical convenience. Speaking of our perception, say visual, auditory, and so on, it is generally said that our perception is more sensitive to the change of stimulus rather than to the absolute value of stimulus. So the vision can be said to be more sensitive to the deviation of a point on a curve rather than the absolute position of the point. Remembering (6), we have

$$z(s+ds)-z(s) = w(s)ds. \tag{14}$$

We can see from this equation that the P-expression w gives the deviation of a point when we proceed unit length along the curve. That's why we use the P-expression w instead of the position z of a point itself.

2. FOURIER DESCRIPTOR

The second transformation to be introduced is the Fourier transformation F which maps the P-expression w to a sequence of complex numbers c(k): for k = 0, ±1, ±2, ···,

$$c(k) = \int_0^1 w(s)\exp[-i2\pi ks]ds. \tag{15}$$

This sequence c(k)'s is called the **spectrum** of the curve z. A number c(k) is also called the **spectral component** at **frequency** k. A subset of the set of Fourier coefficients c(k)'s might have some information with respect to the curve z. Thus it is called **Fourier Descriptors**. In the case of a curve consisting of only line segments (see Fig.5), the coefficient c(k) is easily calculated as follows :

$$\left.\begin{aligned} c(0) &= \sum_{j=1}^{n}(s_j-s_{j-1})\exp[i\theta_{j-1}], \\ c(k) &= \sum_{j=1}^{n}(\exp[i2\pi ks_j]-\exp[i2\pi ks_{j-1}])/2\pi k\exp[i\theta_{j-1}] \quad (k\neq 0). \end{aligned}\right\} \tag{16}$$

Using the spectrum, we may define the **power spectrum** as a sequence of real numbers :

$$p(k) = |c(k)|^2 \quad (k = 0, \pm 1, \pm 2, \cdots). \tag{17}$$

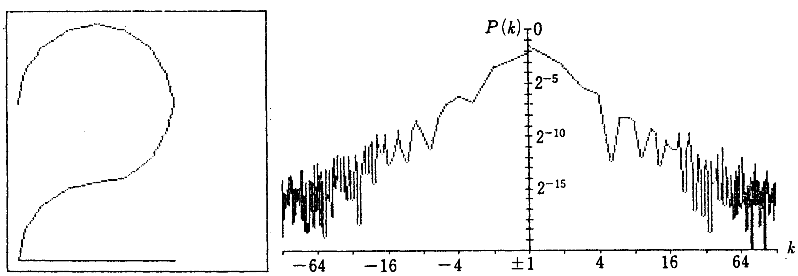

Fig.6 An example of the power spectrum

Fig.6 shows an example of the power spectrum. As seen in this figure, the dominant information of the pattern concentrates on the low frequency part of the power specrum. Thus we can expect to get a rough pattern by means of the spectral components contained in low frequency part.

So the third transformation to be introduced is the **low-pass filter** L_N, which maps the spectrum c to its subset c_N as follows :

$$c_N(k) = \begin{cases} c(k) & (\text{ if } |k| \leq N); \\ 0 & (\text{ otherwise }). \end{cases} \tag{18}$$

Saying in other words, the low-pass filter plays a role to extract the low frequency part of the spectrum. The extracted spectrum c_N is called the **low-pass spectrum** with the cut frequency N.

3. ROUGH TRANSFORMATION

Scine the low-pass spectrum c_N is considered to have the dominant information of a given curve, we might expect to obtain a rough pattern by applying the inverse transformations introduced above to the low-pass spectrum c_N. That is, the rough pattern z_N will be defined as

$$z_N = P^{-1} \cdot F^{-1}(c_N), \tag{19}$$

where F^{-1} and P^{-1} are the Fourier inverse transformation and the inverse of P-transformation, respectively. More concretely, when we write $F^{-1}(c_N)$ as w_N, then

$$w_N(s) = \sum_{k=-N}^{N} c_N(k)\exp[i2\pi ks], \tag{20}$$

and

$$\begin{aligned} z_N(s) &= z(0)+\int_0^s w_N(t)dt \qquad (21) \\ &= z(0)+\sum_{k=-N(\neq 0)}^{N} c_N(k)(\exp[2\pi iks]-1)/2\pi ik. \end{aligned}$$

Now we have finally got the overall transformation :

$$R_N = P^{-1} \cdot F^{-1} \cdot L_N \cdot F \cdot P, \tag{22}$$

which maps a given curve to its rough pattern with the cut freqency N. This transformation is called a **rough transformation.**

Fig.7 shows one of examples of rough patterns obtained by means of the rough transformation. The number N indicates the cut frequency of the low-pass filter used.

As easily seen from this demonstration, the smaller the cut frequency becomes, the larger the degree of roughness becomes. So we may say that the rough transformation introduced here has the ability of good approximation in the sense of visual psychology. This sort of property has been confirmed to appear in the simulation expreiment for a wide range of patterns.

Fig.8 is the case of an open curve. This example seems to indicate the another aspect of the rough transformation. When we draw a pattern, say a chinese character, i.e. Kanji, with a very high speed, we are forced to approximate the given pattern. As a result, we would have an approximated pattern which does not exactly coincide with the original one, but has the same features as the original one. In this situation, the details of the original pattern would be cut out and only its dominant part would be preserved.

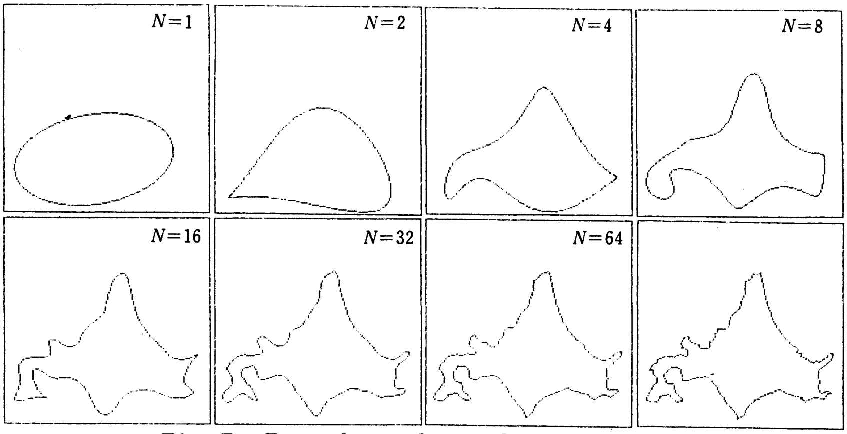

Fig.7 Examples of rough patterns

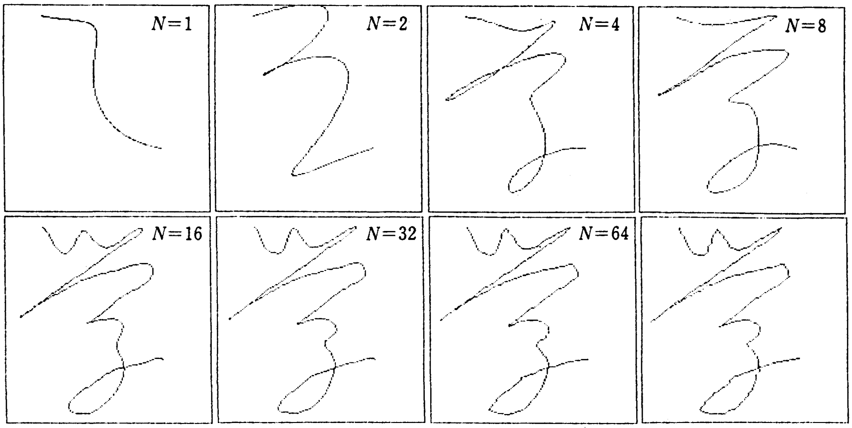

Fig.8 Examples of rough patterns

When we survey those rough patterns from the small N to the large N, we can see that the details which were cut visually out are gradually recovered as the cut frequency N is increasing. Thus, it might be said that the rough transformation also gives a sort of model for a cursive hand-writing or a running style of hand-writing.

5. CONCLUDING REMARKS

When we want to extract automatically a rough pattern from a given pattern, we should set the cut frequency N. As seen in Fig.7 and 8, the proper cut frequency depends on a given pattern. So we should automatically determine the cut frequency as well as the extraction of a rough pattern.

It might be remarked that the notion of the entropy will serve to the automatic selection of the cut frequency. By means of Perseval's equality, it is easily shown that

$$\sum_j p(j) = 1. \tag{23}$$

Hence the power spectrum p satisfies the axiom of probability distribution. We define the complexity of a pattern C having the power spectrum p as follows :

$$G(C) = -\sum_j p(j)\log p(j). \tag{24}$$

Fig.9 shows several examples of the complexity. As easily conjectured, the smaller the complexity becomes, the larger the degree of concentration of the power spectrum becomes.

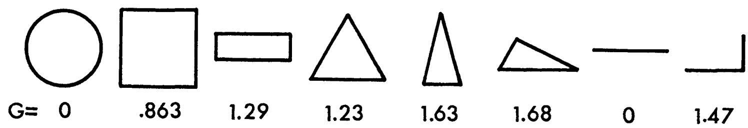

Fig.9 Examples of the complexity

Thus the measure of complexity is considered to serve the determination of the cut frequency. This is one of important problems to be investicated in the near future.

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