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A Stereological Study on Crack Geometry of Discontinuous Rock Masses

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Crack geometry, which is closely related to the mechanical anisotropy of discontinuous materials like rocks and rock masses, can be concisely expressed by a tensor (called the crack tensor) introduced by Oda (1982). In this paper, an actual rock mass (moderately jointed granite) is studied to see if the crack tensor can be actually determined in situ. It is proved that stereology, based on geometrical statistics, provides a sound basis for determining the crack tensor.

INTRODUCTION

Rock masses are commonly treated as isotropic solids in the conventional analyses of rock mechanics. According to the comprehensive review by Gerrard (1977), however, geological material is seldom isotropic. Granite, for example, is not an exception in spite of the isotropic appearance. In fact, the preferred orientation of discontinuities (called cracks) such as microcracks, fissures and joints is universally observed in granite, and it is believed to be one of the major sources controlling the anisotropic, mechanical properties. It is clear that crack geometry (density, size and orientation of cracks) must be considered first when rock masses are treated as anisotropic solids. Oda (1982 & 1984) has introduced a tensor (called the crack tensor) to give a quantitative definition for the crack geometry of discontinuous rock masses. The present purpose is to show that the crack tensor can be actually determined in situ on the basis of stereological consideration.

CRACK TENSOR

Here, "crack geometry" is used to represent a concept concerned with density, size and orientation of related cracks.

1) Density of cracks: If there are $m^{[V]}$ cracks in a statistically homogeneous body of volume V, the crack density ρ is defined as

$$\rho = m^{(V)} / V \tag{1}$$

2) Size of cracks: For simplicity, a crack having area S is replaced by an equivalent circle with a diameter r (i.e., $r=2\sqrt{S/\pi}$). (This is not always a necessary assumption. If the shape of cracks is already known, the typical dimension is used instead of r.) The distribution of crack sizes is then given by a density function f(r) of diameters. It must satisfy

$$\int_0^{r_m} f(r)\,dr = I \tag{2}$$

where r_m is the maximum size of diameters.
3) Orientation of cracks: Orientation of a crack is indicated by two unit vectors, $\underset{\sim}{n}^{[+]}$ and $\underset{\sim}{n}^{[-]}$, normal to the major principal plane (Fig. 1). (Note that $\underset{\sim}{n}^{[+]}$ is parallel, but opposite, to $\underset{\sim}{n}^{[-]}$. Hereafter, $\underset{\sim}{n}$ stands for both $\underset{\sim}{n}^{[+]}$ and $\underset{\sim}{n}^{[-]}$.) A density function $E(\underset{\sim}{n},r)$ is used to represent the statistical distribution of $\underset{\sim}{n}$. It also satisfies

$$\int_0^{r_m} \int_\Omega E(\underset{\sim}{n},r)\,d\Omega dr = I \qquad (3)$$

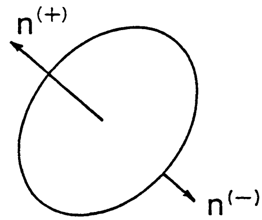

Fig.1 Orientation of a crack

where Ω is a solid angle corresponding to the entire surface of a unit sphere. Here, $E(\underset{\sim}{n},r)$ is symmetric in the sense of $E(\underset{\sim}{n},r)=E(-\underset{\sim}{n},r)$, and it is written as $E(\underset{\sim}{n})\tilde{f}(r)$ if $\underset{\sim}{n}$ and r are statistically independent.

A tensor $\underset{\sim}{F}$ is introduced to give a mathematical definition for the crack geometry in such a manner that all these elements are included (Oda: 1982 & 1984).

$$\underset{\sim}{F} = \frac{\pi\rho}{4}\int_0^{r_m} \int_\Omega r^3 \underset{\sim}{n}\otimes\underset{\sim}{n}\otimes \cdots \underset{\sim}{n} E(\underset{\sim}{n},r)\,d\Omega dr \qquad (4)$$

where $\otimes$ stands for tensor product and the number of $\underset{\sim}{n}$ designates the rank of the tensor.

The reasons that the crack tensor has been accepted as a representative measure of the crack geometry are summarized below:
1) Definite mathematical meaning: $\underset{\sim}{F}$ is a dimensionless tensor, with non-zero components $F_{ij\cdots k}$ $(i,j,\cdots,k=1,2,3)$ only when the rank (=the number of subscripts of $F_{ij\cdots k}$) is even. The components are symmetric in the sense that $F_{ij\cdots k}=F_{ji\cdots k}=F_{kj\cdots i}$. A contraction with respect to any pair of subscripts reduces its rank by 2. The zero-, second- and fourth- rank tensors, for example, are given below by using a fixed orthogonal Cartesian coordinate:

Zero-rank: $$F_0 = \frac{\pi\rho}{4}\int_0^{r_m} r^3 f(r)\,dr$$

Second-rank: $$F_{ij} = \frac{\pi\rho}{4}\int_0^{r_m}\int_\Omega r^3 n_i n_j E(\underset{\sim}{n},r)\,d\Omega dr \qquad (5)$$

Fourth-rank: $$F_{ijkl} = \frac{\pi\rho}{4}\int_0^{r_m}\int_\Omega r^3 n_i n_j n_k n_l E(\underset{\sim}{n},r)\,d\Omega dr$$

$(i,j,k,l=1,2,3)$

where n_i is a direction cosine of $\underset{\sim}{n}$ with respect to the reference axis x_i.
2) Close relation to crack geometry: On the assumption that crack aperture increases in proportion to crack size, Oda (1982) has proved that the scalar F_0 is equivalent to the porosity associated with cracks. Since F_{ij} is a symmetric second-rank tensor just like a stress tensor, one can always find three orthogonal principal axes even though non- orthogonal crack sets are concerned.
3) Close relation to hydro-mechanical properties: In the analyses of hydro-mechanical behaviors of rock masses, engineers must commonly deal with several tensor quantities such as stress and strain. It can be said, therefore, that if the concept of the crack geometry is quantified by a tensor, it is easily taken into account in the analyses. For example, considering elasticity for discontinuous rock masses, Oda & Maeshibu (1984) have formulated an elastic

constitutive equation. Furthermore, the second-rank crack tensor plays an important role when a permeability tensor for jointed rock masses is considered (Oda: 1985).

STEREOLOGICAL STUDY ON CRACK TENSOR

In order to complete the purpose of the paper, the next step is to show the detailed procedure which makes it possible to determine actually the tensor on the basis of field observations. To this end, one assumption is adopted here. That is, the random variables $\underset{\sim}{n}$ and r are statistically independent with one another. Statistically, it means

$$E(\underset{\sim}{n},r) = E(\underset{\sim}{n})f(r) \tag{6}$$

where $E(\underset{\sim}{n})$ and f(r) are the density functions of $\underset{\sim}{n}$ and r respectively. If Eq.(6) cannot be accepted, cracks must be classified into a few homogeneous groups in each of which Eq.(6) is acceptable. In such a case, the individual crack tensors are considered first, and they are summed up afterward.

Orientation of cracks is commonly represented by contour lines of $\underset{\sim}{n}$ on Schmidt's equal area net. Each contour line is labeled by the percentage of concentration of $\underset{\sim}{n}$ per 1% area (e.g., Turner and Weiss: 1963). It is easy to prove that the percentage of each contour line has the same meaning as the contour of $E(\underset{\sim}{n})$ if it is divided by 4π (Oda:1985). Accordingly, the conventional field survey supplies the density function $E(\underset{\sim}{n})$. Great care must be paid, of course, not to be biased when cracks are sampled. On the contrary, it is very difficult to obtain reliable data on the density function f(r) and the crack density ρ.

Using Eq.(6) in Eq.(5), then the crack tensor becomes

$$\begin{aligned} F_{ij\ldots k} &= \frac{\pi\rho}{4}\int_0^{r_m} r^3 f(r)\,dr \int_\Omega n_i n_j \ldots n_k E(\underset{\sim}{n})\,d\Omega \\ &= \frac{\pi\rho}{4}\langle r^3\rangle N_{ij\ldots k} \end{aligned} \tag{7}$$

where

$$N_{ij\ldots k} = \int_\Omega n_i n_j \ldots n_k E(\underset{\sim}{n})\,d\Omega \tag{8}$$

is a symmetric tensor calculated from the density function $E(\underset{\sim}{n})$ only, and

$$\langle r^n\rangle = \int_0^{r_m} r^n f(r)\,dr \tag{9}$$

is an n-th moment of r.

To deal with Eq.(7) further, a straight scanline ab (length h) is set in a rock mass as being parallel to a unit vector $\underset{\sim}{q}$ (Fig.2). A column with the following characters is considered: The central axis coincides with the scanline. The upper and lower planes consist of $(\underset{\sim}{n},r)$-cracks so that the cross-sectional area equals $(1/4)r^2|\underset{\sim}{n}.\underset{\sim}{q}|$ (=the area of an $(\underset{\sim}{n},r)$-crack projected on the plane perpendicular to $\underset{\sim}{q}$). It is easily seen that any $(\underset{\sim}{n},r)$-cracks must cross the scanline if the centers are placed just inside the

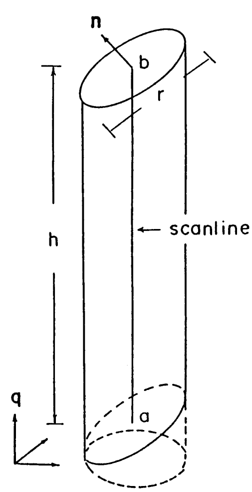

Fig.2 Column parallel to $\underset{\sim}{q}$

column.

The length h of the scanline is so long that there are many cracks inside the volume V . Note that multiplying the volume by ρ yields the total number of the cracks whose centers are located inside the column:

$$\rho V = \frac{\pi\rho}{4} h r^2 |\underset{\sim}{n}.\underset{\sim}{q}| \tag{10}$$

If ρV is further multiplied by the rate of $(\underset{\sim}{n},r)$-cracks, it becomes the number $dN^{[q]}$ of $(\underset{\sim}{n},r)$-cracks whose centers are located inside the volume;

$$dN^{[q]} = \frac{\pi\rho}{4} h r^2 |\underset{\sim}{n}.\underset{\sim}{q}| 2E(\underset{\sim}{n}) f(r) d\Omega dr \tag{11}$$

Of course, $dN^{[q]}$ corresponds to the number of $(\underset{\sim}{n},r)$-cracks crossed by the scanline. In order to count all cracks crossed by the scanline, Eq.(11) must be integrated over $\Omega/2$ and $0 \leqq r \leqq r_m$ with the following result:

$$\begin{aligned} \frac{N^{[q]}}{h} &= \frac{\pi\rho}{4} \int_0^{t_m} r^2 f(r) dr \int_{\Omega/2} |\underset{\sim}{n}.\underset{\sim}{q}| 2E(\underset{\sim}{n}) d\Omega \\ &= \frac{\pi\rho}{4} < r^2 > < |\underset{\sim}{n}.\underset{\sim}{q}| > \end{aligned} \tag{12}$$

where

$$< |\underset{\sim}{n}.\underset{\sim}{q}| > = \int_\Omega |\underset{\sim}{n}.\underset{\sim}{q}| E(\underset{\sim}{n}) d\Omega \tag{13}$$

is a scalar calculated from the density function $E(\underset{\sim}{n})$. Here, $N^{[q]}/h$ is the number of cracks crossed by the unit length of the scanline in the direction of $\underset{\sim}{q}$, and $<|\underset{\sim}{n}.\underset{\sim}{q}|>$ is a correction factor with respect to the selected direction $\underset{\sim}{q}$. That is, $N^{[q]}/h$ divided by $<|\underset{\sim}{n}.\underset{\sim}{q}|>$ must be constant irrespective of the direction $\underset{\sim}{q}$.

Actual cracks are observed as trace lines on cliffs or excavated walls. Let $\psi(t)$ be a density function for the distribution of trace lengths t. The stereological study by Oda (1984 & 1985) has proved that $\psi(t)$ is uniquely determined, not depending on the orientation of the observed wall, if $\underset{\sim}{n}$ is a statistically independent variable of r. Besides, the n-th moment of t is related to the moments of r as follows (also see, Kendall and Moran: 1963);

$$< t^n > = \frac{< r^{n+1} >}{< r >} \int_0^{\pi/2} \sin^{n+1}\theta d\theta \tag{14}$$

where

$$< t^n > = \int_0^{t_m} t^n \psi(t) dt \tag{15}$$

Using Eq.(14), together with Eq.(12), $<r^3>$ becomes

$$< r^3 > = \frac{3 < t^2 > N^{(q)}/h}{2\rho < t > < |\underset{\sim}{n}.\underset{\sim}{q}| >} \tag{16}$$

Substituting Eqs.(12) and (16) in Eq.(7), the crack tensor is finally given by

$$F_{ij\ldots k} = \frac{3\pi}{8} \frac{< t^2 >}{< t >} \frac{N^{[q]}/h}{<|\underset{\sim}{n}.\underset{\sim}{q}|>} N_{ij\ldots k} \tag{17}$$

AN EXAMPLE

In order to exemplify the detailed procedure leading to the determination of crack tensor, a typical site is chosen to investigate. (Here, only the second rank tensor is calculated. Note, however, that no difficulty arises in the calculation of higher rank tensors.) The site, located near Nakatsugawa, Central Japan, is composed of moderately jointed, fresh granite. Joints were surveyed with the special emphasis on the following points:

1) Orientation of joints: Three orthogonal scanlines (EW, NS and vertical) were set on the surface of the granite (25mx20mx7m). Strik and dip were measured whenever the scanlines cross a joint. Orientation of joints is shown by plotting their normals as poles on Schmidt's equal area net (Fig.3). In regard to Fig.3, all joints were classified into three groups ((A),(B) and(C)). The density function $E(\underset{\sim}{n})$ for each group is shown separately in Fig.4. Using the data of Fig.4 in Eq.(8), $N_{ij}^{[A]}$, $N_{ij}^{[B]}$ and $N_{ij}^{[C]}$ are calculated as follows:

$$N_{ij}^{[A]} = \begin{bmatrix} N_{11} & N_{12} & N_{13} \\ N_{21} & N_{22} & N_{23} \\ N_{31} & N_{32} & N_{33} \end{bmatrix} = \begin{bmatrix} 0.135 & -0.010 & -0.041 \\ -0.010 & 0.071 & 0.078 \\ -0.041 & 0.078 & 0.794 \end{bmatrix}$$

$$N_{ij}^{[B]} = \begin{bmatrix} 0.789 & 0.049 & 0.123 \\ 0.049 & 0.058 & 0.011 \\ 0.123 & 0.011 & 0.153 \end{bmatrix}; \quad N_{ij}^{[C]} = \begin{bmatrix} 0.031 & -0.067 & -0.003 \\ -0.067 & 0.994 & 0.080 \\ -0.003 & 0.080 & 0.025 \end{bmatrix} \tag{18}$$

In the calculation, we referred to the axes x_1, x_2 and x_3 given in Fig.4.

2) Trace lengths of joints: Joint traces, which were visible on the horizontal section (25mx20m), were carefully mapped (Fig.5). Two data were made using the map of the joint traces for each group separately: Firstly, a scanline pointing to a direction $\underset{\sim}{q}$ was set. The number of cracks crossed by the scanline was counted to give $N^{(q)}/h$, and the correction term $\langle|\underset{\sim}{n}.\underset{\sim}{q}|\rangle$ was also calculated by using the density function of Fig.4. Several trials have proved that $(N^{[q]}/h) / \langle|\underset{\sim}{n}.\underset{\sim}{q}|\rangle$ remains almost constant, not depending much on the selected direction $\underset{\sim}{q}$ of the scanline. Secondly, the frequency histograms of the trace lengths were prepared (Fig.6). The joints belonging to group (C) do not appear on the horizontal map because they are subparallel to the observed plane. So, two large vertical cliffs located near the horizontal section were carefully sketched to provide the same histogram (Fig.6). These histograms are similar in the shape, but differ in the mean and standard deviation. This is the main reason that the joints were classified into the three groups. Using these diagrams, the moments of the trace length, $\langle t \rangle$ and $\langle t^2 \rangle$, were calculated.

Using all these data in Eq.(17), the crack tensors $F_{ij}^{[A]}$, $F_{ij}^{[B]}$ and $F_{ij}^{[C]}$ for the groups (A), (B) and (C) were separately calculated, and were summed up to give the final crack tensor F_{ij}, as follows:

$$F_{ij} = \begin{bmatrix} 9.965 & 0.162 & 1.651 \\ 0.162 & 5.091 & 1.066 \\ 1.651 & 1.066 & 7.715 \end{bmatrix} \tag{19}$$

If the reference axes are selected as the principal axes x_1', x_2' and x_3' of the crack tensor, then Eq.(19) becomes

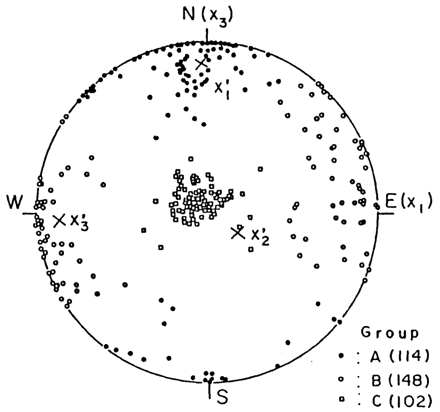

Fig.3 Schmidt's equal area projection of poles normal to joints (lower hemisphere), with the principal axes of F_{ij} .

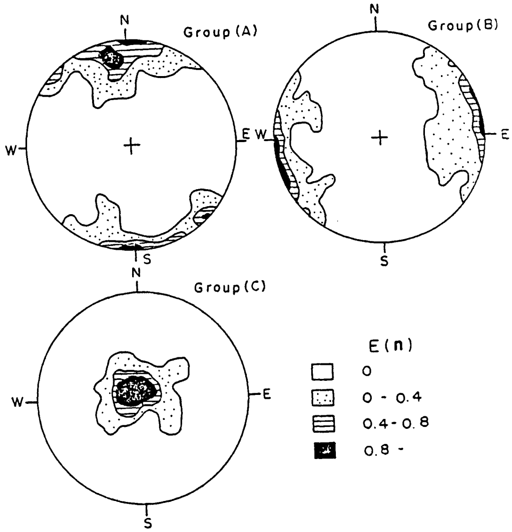

Fig.4 Density function $E(\underset{\sim}{n})$ for each joint group.

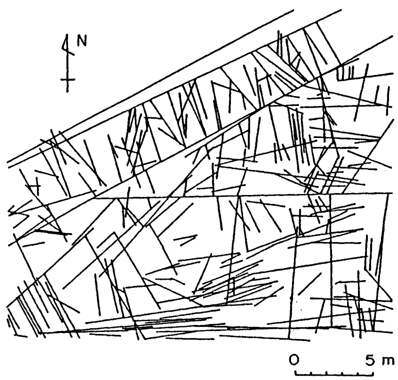

Fig.5 Map showing the joint traces on a horizontal section

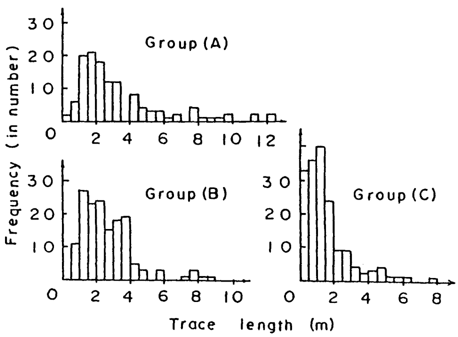

Fig.6 Histograms of the trace lengths. (Four more joints having 13.9m, 21.0m, 23.7m and 28.8m in trace length must be added to the diagram for group (A).)

$$F_{ij} = \begin{bmatrix} 10.919 & 0 & 0 \\ 0 & 7.282 & 0 \\ 0 & 0 & 5.169 \end{bmatrix} \quad (20)$$

with the principal axes plotted on Fig.3.

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4-2

Q: Could you explain the relation between two dimensional fabric tensor and a three dimensional one in the sence of image analysis? (M. Satake)

A: In Rock Mechanics, we are mainly concerned with the three-dimensional structure formed by geological discontinuities. The corresponding fabric tensor which is expressed by a three-dimensional (or one dimensional) observations using rock exposures and drill holes.

Stereological study which is based on geometrical statistics has shown that the fabric tensor in three-dimension can be expressed in terms of 1) the number of cracks crossed by a unit length of a scanning line, 2) number of cracks associated with a unit area of a scanning plane and 3) the density function which describes the orientation of crack normal unit vectors.