*Science on Form: Proceedings of the First International Symposium for Science on Form,*
General Editor: S. Ishizaka, Editors: Y. Kato, R. Takaki, and J. Toriwaki, 147–155.


# Spatial Tessellations and Their Stereology

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Keywords: spatial tessellations of polyhedra; point, line and plane processes in space; stereology

First, in this review paper, basic stereological estimators for spatial tessellations are presented. Then various specific tessellation models are introduced, based upon Poisson flat processes of differing dimensions in $R^3$ Stereological aspects of these models are considered.

## INTRODUCTION

Classical stereology relates to the estimation of the spatial structure of irregular materials, on the basis of plane and line sections. Supposing the material to be deterministic, sections must be taken random in a precisely defined way, whereupon a class of stereological estimators of tessellation characteristics emerges. Often, however, alternative prior knowledge of the material is available, and then a more powerful approach may be to formulate a parametric class of homogeneous (and maybe isotropic) spatial stochastic models for the material, based upon this knowledge; for example, by taking a specific parametric class of covariance functions. In this case, sections need only be arbitrary, i.e. chosen independently of knowledge of the constitution of the specimen. The classic stereological estimators extend to this case, but with differing variance expressions. However, the more detailed and accurate the model, the more accurate the resulting estimators may be expected to be.

## STEREOLOGY OF AGGREGATES AND TESSELLATIONS

Write $X$ for a bounded domain of three-dimensional euclidean space $R^3$, and consider a disjoint aggregate $\{X_i\}$ $(i = 1,2,...)$ of such domains $(X_i \cap X_j = \emptyset$ whenever $i \neq j)$. We say $\{X_i\}$ is *space-filling* in $R^3$ if $\cup X_i = R^3$. Writing $K$ for a convex domain, it is clear that if disjoint $\{K_i\}$ is space-filling in $R^3$, then all interfaces between pairs of $K_i$ are planar, so that all $K_i$ are convex polyhedra. Thus, writing $P$ instead of $K$, we say $\{P_i\}$ is a *tessellation* of $R^3$ or, alternatively, $\{P_i\}$ *tessellates* $R^3$. The $P_i$ are the *cells* of the tessellation.

To be able to practice stereology, we need to precisely define certain basic types of random plane (and line) sections of bounded domains $X$ in $R^3$. Consider the following stochastic construction:

(i) Contain $X$ in a sphere $Q$ (preferably of as small a radius as possible).

(ii) Take a uniform random (i.e. isotropic) direction $u$ emanating from the centre of $Q$.

(iii) Take a uniform random point $z$ on the diameter of $Q$ in direction $u$.

(iv) Construct the plane through $z$ orthogonal to $u$.

(v) If this plane intersects $X$, then call it $T_2$. If not, then repeat (i)-(iv) until $X$ is hit by $T_2$. (It is now clear why $Q$ should be chosen small!)

$T_2$ is an *IUR* (isotropic uniform random) plane section of $X$, with the following key property: for $Y \subset X$,

$$Pr(T_2 \cap Y \neq \emptyset) = M_1(Y)/M_1(X) , \quad (1)$$

where $M_1$ denotes 'mean caliper diameter'.

For $T_2$ IUR through $X$, the perimeter $B(T_2 \cap X)$ of the section is a random variable. By geometrical probability/ integral geometry, its mean value is

$$E\{B(T_2 \cap X)\} = (\pi/4)\ S(X)/M_1(X) , \quad (2)$$

where $S$ denotes surface area (Miles & Davy, 1976). (2) is stereological in the sense that, if repeated IUR sectioning is feasible, the *spatial* ratio $S(X)/M_1(X)$ may be estimated on the basis of *planar* measurements.

Next consider a particle aggregate $\{X_i\}$ $(i = 1,...,N) \subset$ bounded domain $X$. Its *density* $N_V$ is $N/V(X)$. Aggregate means are denoted by a bar, e.g. $\overline{V^2S} = \Sigma_1^N V(X_i)^2 S(X_i)/N$. For an IUR planar section $T_2$ of $X$, (1) and (2) imply

$$E(\Sigma_1^n B_j) = (\pi/4)\Sigma_1^N S(X_i)/M_1(X) \quad (3)$$

$$E(n) = \Sigma_1^N M_1(X_i)/M_1(X) \quad (4)$$

where $B_j$ $(j = 1,...,n)$ are the perimeters of the non-void $T_2 \cap X_j$. 'Dividing' (3) by (4) suggests that $\bar{B} = \Sigma_1^n B_j/n$ may be used to estimate $(\pi/4)\overline{S}/\overline{M_1}$. In fact, this is true either

(i) in the limit of repeated IUR sections, or

(ii) for an unbounded stochastically homogeneous and isotropic random aggregate in $R^3$, where $T_2$ need only be arbitrary (Miles, 1978). 'Homogeneous' and 'isotropic' here and elsewhere relate to processes stochastically invariant under arbitrary translations and rotations, respectively.

IUR line sections $T_1$ of X may be defined in analogous fashion. Basic stereological estimators for IUR sections are as follows:

| <u>plane sections</u> | <u>line sections</u> | |
|---|---|---|
| $E(N_A) = N_V \bar{M}_1$ | $E(N_L) = N_V \bar{M}_2$ | |
| $E(\bar{\bar{C}}) = \bar{K}/\bar{M}_1$ | $E(\bar{\bar{\nu}}) = (1/4)\bar{S}/\bar{M}_2$ | |
| $E(\bar{\bar{B}}) = (\pi/4)\bar{S}/\bar{M}_1$ | $E(\bar{\bar{L}}) = \bar{V}/\bar{M}_2$ | (5) |
| $E(\bar{\bar{M}}_1^{(2)}) = \bar{M}_2/\bar{M}_1$ | $E(\overline{\overline{[L^4]}}) = (3/\pi)\overline{V^2}/\bar{M}_2$ | |
| $E(\bar{\bar{A}}) = \bar{V}/\bar{M}_1$ | | |
| $E(\bar{\bar{\Theta}}) = (1/2\pi)\overline{V^3}/\bar{M}_1$ | | . |

For further details, the reader may consult Miles (1985).

Thus the following may be estimated:

By plane or line sections: $N_V\bar{V}$, $N_V\bar{S}$, $N_V\bar{M}_2$

By plane sections only: $N_V\bar{M}_1$, $N_V\bar{K}$, $N_V\overline{V^3}$

By line sections only: $N_V\overline{V^2}$ .

Unfortunately, no single characteristic may be estimated. Note that $N_V\bar{V} = 1$ for a tessellation. In particular, $\overline{V^2}/\bar{V}$, $\bar{V}^3/\bar{V}$ and $\overline{V^3}/\overline{V^2}$ may all be estimated, i.e. $\bar{V}$ and $\overline{V^2}$ for the V-weighted aggregate distribution, and $\bar{V}$ for the $V^2$-weighted distribution. Thus, especially in the case where alternative knowledge of the aggregate may be available, parametric families of V distributions may be fitted to data (Miles, 1985).

In the case of a homogeneous and isotropic random specimen, the above estimators involving $\bar{V}$, $\overline{V^2}$ and $\overline{V^3}$ have a strikingly convenient practical form, which has been developed and described by Gundersen & Jensen (1985).

Stereology requires that either the specimen or the sections be isotropic. In case (i) above the sections are isotropic, and in case (ii) the specimen is isotropic.

Should the random specimen be only homogeneous, then isotropic sectioning will be necessary. Practical approximations to this involve taking equal area plane sections with orientations those of the faces of, for example, a regular dodecahedron.

In the case of tessellations, the interfacet structure, i.e. the relations between neighbouring cells, is of interest. As an example, we may observe from plane sections the distribution of the number of planar cells having the same vertex. Clearly, this estimates the edgelength-weighted distribution of the number of spatial cells having the same edge.

## TESSELLATIONS FROM RANDOM PLANES

Clearly a collection of 'random' planes in $R^3$ has the effect of partitioning $R^3$ into an aggregate of random convex polyhedra. A simple case is that in which the random planes have just three orientations - orthogonal to Ox, Oy and Oz, their intersections with these axes being stochastic point processes $\{x_i\}$, $\{y_j\}$ and $\{z_k\}$. If these processes are independent, and each is homogeneous, then the resultant tessellation is also homogeneous, but clearly not isotropic. However, the next example is isotropic.

Specify a plane in $R^3$ by its orientation, i.e. a unit vector u normal to it, $\in$ some hemisphere H of directions; and by its $\pm$ perpendicular distance p from O in the direction u . Now we can define the random plane process $\Pi_\rho(2,3)$ as $\{(p_i,u_i)\}$ $(i=0,\pm1,\ldots)$, where : $\{p_i\}$ , $\{u_i\}$ are independent; $\{p_i\}$ are the points of a homogeneous Poisson point process $P_\rho(0,1)$ of intensity $\rho$ on a line (with, e.g., $p_0$ defined by $|p_0| \leq$ all other $|p_i|$); and $\{u_i\}$ are IID uniform on H . Then $\Pi_\rho(2,3)$ is homogeneous and isotropic in $R^3$ . [In fact, by work of Kallenberg (1977) and others it appears that, for a plane process in $R^3$ to be homogeneous and isotropic, it must essentially be of $\Pi_\rho(2,3)$ type.] A basic hitting property is that, for bounded domains $\subset R^3$ , the number of planes of $\Pi_\rho(2,3)$ hitting X has a Poisson $\{\rho M_1(X)\}$ distribution; moreover, given that the number of hitting planes is N , they are independent IUR planes through X .

In fact, $\Pi_\rho(2,3)$ and $\Pi_\rho(0,1)$ are examples of $\Pi_\rho(s,d)$, *Poisson* s-*flats in* $R^d$ $(0 \leq s < d < \infty)$, where $R^0$ = 0-flat = point, $R^1$ = 1-flat = line, $R^2$ = 2-flat = plane, $R^3$ = 3-flat = 'solid',... . The *intensity* $\rho$ is the mean s-content of s-flat per unit d-volume (Miles, 1971, (3.19C)). The hyperplane processes $\Pi_\rho(d-1,d)$ tessellate $R^d$ into random convex polytopes $(d=1,2,\ldots)$ .

We write $P$ for the random tessellation determined by $\Pi_\rho(2,3)$ . Its facet structure is such that each face belongs to two cells (a universal tessellation property), each edge belongs to four cells and each vertex belongs to eight cells. The cell volume distribution is unknown, unlike the inradius distribution, which is exponential $(2\rho)$ . Another class of known distributions are the $M_1$-values of the subaggregate of cells with $N_2$ faces, which is $\Gamma(N_2-3,\rho)$. Various 1st and 2nd order moments of cell characteristics, including the first three order moments of V , are known. For further details, see Miles (1972b).

The stereology of $\Pi_\rho(2,3)$ is very simple, since (Miles, 1971, (3.29T))

$$T_2 \cap \Pi_\rho(2,3) = \Pi_{\pi\rho/4}(1,2) \ ,$$

$$T_1 \cap \Pi_\rho(2,3) = \Pi_{\rho/2}(0,1) \ .$$

*Anisotropic case.* Taking $\{u_i\}$ in the above stochastic construction of $\Pi_\rho(2,3)$ as IID on H , with common generalized (i.e., including atoms) probability density b(u), yields homogeneous but anisotropic ($\equiv$ not isotropic) Poisson planes in $R^3$ . b is its orientation density, but a more useful function is $\tau(u) = \rho b(u)$ (so that $\rho = \int_H \tau(u)du$ and $b(u) = \tau(u)/\int_H \tau(u)du$ . Then we write $\Pi_\tau(2,3)$ and again $\mathcal{P}$ for the generated tessellation. Note the 'three orientations' and 'isotropic' cases are both special cases.

The stereology becomes more interesting. Thus a line with direction $v$ intersects $\Pi_\tau(2,3)$ in a $\Pi_{\rho(v)}(0,1)$, where

$$\rho(v) = \int_H |u.v| \; \tau(u)du \qquad (v \in H). \tag{6}$$

Should $\rho(v)$ be known (i.e., estimated with sufficient accuracy), then (6) constitutes an integral equation for $\tau$ . Its solution appears to be an open problem. Next consider plane sections. A plane $T_2(v)$ with normal $v(\in H)$ intersects $\Pi_{\tau_3}(2,3)$ in a $\Pi_{\tau_2(v)}(1,2)$ where, if $(p,\theta)$ parametrizes lines in $T_2(v)$ ,

$$\tau_2(p,\theta;v) = \int_{0 \;\; u\perp(p,\theta)}^{\pi} \tau_3(u) \; \sin^2\phi \; d\phi \tag{7}$$

where $0 < \phi \equiv \cos^{-1}(u.v) < \pi$ . Integration of (7) yields

$$\int_{0 \;\; v\perp(p,\theta)}^{\pi} \tau_2(p,\theta;v)dv = (\pi/2) \int_{0 \;\; u\perp(p,\theta)}^{\pi} \tau_3(u)du . \tag{8}$$

Should the left side of (8) be known (or estimable with sufficient accuracy), then the right side - the integrated great circle sections of $\tau_3$ - is known, and so $\tau_3$ may be determined by standard Radon theory. For a discussion of other sectional aspects, see Matheron (1974). Finally, it should be noted that (6), (7) apply more generally to the induced orientation distribution of a homogeneous random surface in $R^3$ (Miles, 1972a, Section 2). Notwithstanding the above, isotropic sectioning gives rise to the estimators (5) of tessellation means.

*Nested Poisson tessellations.* An interesting variation on the above tessellations are nested Poisson tessellations, an example of which was introduced by Serra (1982, p.296). Thus suppose $\Pi_{\tau_1}(2,3)$ determines $\mathcal{P}_1$ as above. For each cell $c$ of $\mathcal{P}_1$ , generate an (independent) realization of $\Pi_{\tau_2}(2,3)$ , and retain its intersection with $c$ . In this way we get the *nest* $\Pi_{\tau_1,\tau_2}(2,3)$ , which determines the tessellation $\mathcal{P}_{\tau_1,\tau_2}$ . Then

(i) The cell distributions of $\mathcal{P}_{\tau_1,\tau_2}$ are the same as those of $\mathcal{P}_{\tau_1+\tau_2}$ .

(ii) If $T_2 \cap \Pi_{\tau_i}(2,3) = \Pi_{\sigma_i}(1,2)$ $(i=1,2)$ , then

$$T_2 \cap \Pi_{\tau_1,\tau_2}(2,3) = \Pi_{\sigma_1,\sigma_2}(1,2) .$$

n-fold nests $\Pi_{\tau_1,\ldots,\tau_n}(d-1,d)$ have corresponding properties. Such nests clearly have wider modelling potential than the 'straight' Poisson model.

*Poisson laminae.* Next consider processes of interpenetrating laminae (thick planes) in $R^3$. A lamina may be specified by its mid-plane $(p,u)$ together with its semi-thickness $w$. Thus suppose $b(u)$ is replaced by a (generalized) joint probability density $b(u,w)$, having marginal density $b(u)$ (on $H$), and marginal expectation $E(w)$. Poisson laminae $\Pi_{\tau(u,w)}(2,3)$ are defined in the usual way, i.e. $\{p_i\}$ as above, with $\{u_i,w_i\}$ conforming to $b(u,w)$. For it,

$$\Pr(\text{arbitrary point of } R^3 \text{ lies in no lamina})$$

$$= \exp\{-2\rho E(w)\} = \text{uncovered fraction of } R^3 .$$

Thus, if $E(w) < \infty$, the interstices between the laminae form a polyhedron aggregate $\mathcal{P}_{\tau(u,w)}$. Remarkably, the cells of $\mathcal{P}_{\tau(u,w)}$ have the same distributions as those of $\mathcal{P}_{\tau(u)}$ (Miles, 1961). Stereologically, plane and line sections are corresponding Poisson strip and interval processes.

## TESSELLATIONS FROM RANDOM POINTS

*Voronoi tessellations.* Now we consider the other main source of specific random tessellations, viz. those determined by the Voronoi operation on collections $C$ of (point) *particles* in $R^3$. Thus let $C = \{x_i\}$, supposed for convenience to be in mutual general position. Each point $x \in R^3$ has a nearest particle, $n(x)$ say. $\{y : n(y) = n(x)\}$ is a convex polyhedral cell, containing $n(x)$ as *nucleus*. The aggregate of such cells is the *Voronoi* (sometimes Dirichlet, or Thiessen) *tessellation* $V$ with respect to $C$. Each cell edge $\in$ three cells of $V$ (the edge is a portion of the 'circumline' of the three corresponding particles), and each vertex $\in$ four cells (being the circumcentre of the four corresponding particles). We call such tessellations *normal*, since they are the ones most frequently encountered in practice. Many mean values of $V$ with respect to $\Pi_\rho(0,3)$, the simplest particle process, are known : polyhedron means, means for edges and planar faces, and stereological means for plane and line sections (see Miles, 1972b, Section 5). Since $\Pi_\rho(0,3)$ is isotropic, so too is $V$ with respect to it. Anisotropic tessellations result from anisotropic $C$, or by affine transformation of isotropic tessellations.

*Generalized Voronoi tessellations.* In fact, $x$ above has not only a nearest particle, but a set of $n$ nearest particles $(n=1,2,\ldots)$. $\{y$ : same set of $n$ nearest particles as $x\}$ is again a convex polyhedral cell. The aggregate of such cells is the generalized Voronoi tessellation $V_n$ with respect to $C$, which is, like $V = V_1$, normal. The planar version has been considered in some detail by Miles (1970, Sections 7,10), Miles & Maillardet (1982) and Maillardet (1982). One elementary property may be noted : since there is a one-to-one correspondence between the cells of $V_2$ and the plane faces of $V$,

$$E_2(V) = 2\ E_1(V)/E_1(N_2)\ . \tag{9}$$

*Skeletonized tessellations.* Consider a convex polyhedron $P$, and label its plane faces $F_i$ $(i = 1,\ldots,N_2)$. Write $P_i(F_i)$ for the sub-polyhedron of $P$ comprising points whose nearest face is $F_i$. Thus $P = \cup_1^{N_2} P_i$. The union of the common boundaries of pairs of the $P_i$ is the skeleton of $P$ (Serra, 1982, p.375). Next consider a homogeneous random tessellation $\mathcal{P}$, with aggregate $\mathcal{F}$ of plane faces. Associate with each $F \in \mathcal{F}$ the polyhedron union $P(F)$ of the two polyhedra $P'(F)$, $P''(F)$ on its two sides. The aggregate of such $P(F)$ tessellates $R^3$, and may be called the skeletonized tessellation $\mathcal{S}(\mathcal{P})$. Property (9) extends, on replacing 1,2 by $\mathcal{P}$, $\mathcal{S}(\mathcal{P})$ respectively. Note that the Voronoi rule 'nearest edge' applied to the edges of $\mathcal{P}$ does not in general yield a *polyhedral* tessellation. Note also that skeletons are based on plane bisectors of *plane-pairs,* whereas $\{\mathcal{V}_n\}$ are based on plane bisectors of *point-pairs.*

*Delaunay tetrahedral tessellations.* Each vertex of $\mathcal{V}$ is vertex of four cells of $\mathcal{V}$. Consider the tetrahedron formed by the corresponding four nuclei. The aggregate of such tetrahedra is a tessellation! - the Delaunay tessellation $\mathcal{D}$ with respect to C (Rogers, 1964). (In fact $\mathcal{D}$ tessellates the convex hull of C.) $\mathcal{D}$ may be regarded as dual to $\mathcal{V}$.

For $\Pi_\rho(0,3)$ particles, the (ergodic) distribution of these tetrahedra is known : if the four particle vertices with respect to their circumcentre are $ru_1$, $ru_2$, $ru_3$ and $ru_4$, then the joint density is

$$f(r;u_1,u_2,u_3,u_4) \propto \exp(-4\pi\rho r^3/3)\ r^8\ \Delta(u_1,u_2,u_3,u_4)\ ,$$

where $\Delta(u_1,u_2,u_3,u_4)$ is the volume of the tetrahedron with vertices $u_1,u_2,u_3,u_4$ (Miles, 1974). Stereologically, plane sections of $\mathcal{D}$ are unions of triangles and convex quadrilaterals. Similarly, a $T_3$ section of $\mathcal{D}$ in $R^4$ is a tessellation comprising 4- and 5-faced polyhedra. Planar sections of the latter are unions of 3-, 4- and 5-gons.

*Sectional Voronoi tessellations.* Let us write $\mathcal{V}(d)$ for $\mathcal{V}$ in $R^d$ with respect to $\Pi_\rho(0,d)$ $(d = 1,2,\ldots)$. First note $T_2 \cap \mathcal{V}(3)$ is a planar tessellation topologically similar to $\mathcal{V}(2)$. Similarly

$$T_3 \cap \mathcal{V}(4) = \text{spatial normal tessellation}\quad \mathcal{V}(3,4)$$

$$\vdots$$

$$T_3 \cap \mathcal{V}(d) = \text{spatial normal tessellation}\quad \mathcal{V}(3,d)\ .$$

$\{\mathcal{V}(3,d)\}$ $(d = 3,4,\ldots)$ constitute a parametric class of random tessellations of $R^3$, with $\mathcal{V} = \mathcal{V}(3) = \mathcal{V}(3,3)$. By construction, they have the stereological property

$$T_t \cap \mathcal{V}(s,d) = \mathcal{V}(t,d) \qquad (t < s)\ .$$

It may be shown that the mean number of plane facets, $E(N_2)$, takes the value 15.54 for $V(3,3)$ and $\to 13.39$ for $V(3,d)$ as $d \to \infty$ (Miles, 1972b, 1984). Hence, since empiric studies of normal tessellations occurring in nature commonly possess values of $E(N_2)$ in the range 13 to 15, $\{V(3,d)\}$ may offer excellent models of such phenomena. In fact, the stochastic construction of $V(3,d)$ may be restricted to four dimensions, by the following lemma (Miles, 1972b). $V(3,d)$ is stochastically equivalent to the intersection of the 3-flat $x_4 = 0$ in $R^4$ with the $V(4)$ in $R^4$ with respect to an *inhomogeneous* Poisson particle process with non-constant intensity $\rho(x_1,x_2,x_3,x_4) \propto x_4^{d-4}$.

*Tanemura & Hasegawa's model.* For many observed tessellations, the cells of the above Voronoi models are too variable to allow good modelling. One solution is to take the underlying particle process in $R^3$ to be more regular than $\Pi_\rho(0,3)$, e.g. as a soft- or hard-core particle process. An alternative approach, carried out in the plane by Tanemura & Hasegawa (1980), is to begin with the standard Voronoi; to replace the nuclei by new nuclei located at the centroids of the vertices of each cell; to form a new Voronoi with respect to these new nuclei; and to sequentially repeat this procedure. The procedure tends to equalize the sizes of neighbouring cells, and in the plane Tanemura & Hasegawa have shown that in the limit, after several hundred repetitions, random tessellations of remarkably equi-sized polygonal cells result. The limits show no overall favoured orientations - only locally favoured orientations, unlike hexagonal honeycomb tessellations. It is to be hoped that sometime the procedure will be carried out in $R^3$.

## TESSELLATIONS FROM RANDOM LINES

Since the above tessellations stem from bases $\Pi_\rho(2,3)$ and $\Pi_\rho(0,3)$, it may be asked whether further tessellations stem from $\Pi_\rho(1,3)$ - 'random lines in space'. For it, the standard Voronoi construction, by 'nearest line', partitions $R^3$ into space-filling tubes surrounding the lines of $\Pi_\rho(1,3)$, with piecewise hyperboloid boundaries. The mean cross-sectional areas of these tubes is $1/\rho$. Conventional (polyhedral) tessellations may be obtained by constructing a $\Pi_\lambda(0,1)$ of collinear particles on each line of $\Pi_\rho(1,3)$, and forming the usual $V$ with respect to the resulting totality of particles.

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3-5
C: In the first part of your presentation, i.e. estimations of stereological quantities of tessellations does in fact not rely on convex cells in the tessellations. (H. J. Gundersen)

Q: I do not really understand the significance of voronoi-tessellations in stereological applications. Could you give some comments on its possible usefulness? (G. Bernroider)

A: 3-dimensional sectors of $\delta$dimensional voronoi tessellations may well offer very good statistiscal models for plant cell structure, metal granular structure, etc. (see my paper)
The value of such statistical models in stereology is that a prior information about the material being considered may be incorporated, with a resulting substantial improvement in the stereological estimates finally obtained.