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Directed Dendritic Fractals

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We propose a model of random dendritic structures, which is based on random aggregation of flows in d-dimensional space. This model, in the case d=2, becomes Sheidegger's model of rivers, and in the case d=3 it may be regarded as a model of blood vessels. Random dendritic structures numerically produced by this model are found to be fractals, and their fractal dimensions are estimated as 1.50, 1.85, 1.96 and 2.00 for d=2,3,4 and ≥ 5, respectively. Size distribution of the fractals obeys a power law and the whole space is packed densely with those dendritic structures.

§1 Introduction

In nature, we can find a variety of complicated dendritic structures such as lightnings, rivers, blood vessels, cracks, etc. Although each of them belongs to different field of science, their geometrical structures look very similar. There might be a common origin.

About 30 years ago, T. Okamoto developed a theory of dendroid system (Okamoto:1951). He considered conservative flows which have the nonlinear property that flow efficiency is higher for stronger flows. In such situation, flows are apt to join together in order to make the total flow efficiency higher. By applying variation principle, he showed that the flow pattern which maximizes the total efficiency is dendritic. His results are worthy of attention, however, he failed to obtain realistic dendritic structures because no randomness is taken into account in his model.

Recently, one of the authors has studied the pattern formation process of dendritic structures in brittle fracture and electric breakdown (Takayasu:1985), and the following fact is clarified. In these problems, tensions in brittle fracture and electric currents in electric breakdown can be regarded as the conservative flows, namely, they satisfy the conservation law. The elementary process of fracture and electric breakdown play the role to join the flows. In the case that there are random fluctuations in some field variables such as rigidity or conductivity at the initial state, the flows gradually join randomly and form a dendritic structure at the end. It is very interesting that nonlinearity of flow efficiency introduced by Okamoto is not neccesary in these cases. Instead of it, other mechanisms which combine the flows successfully create dendritic structures. From this result, we may conjecture that the most important point for the dendritic structures is the simple aggregation process of conservative flows.

In the next section, we propose a very simple model of random dendritic structure based on the above idea. River-like dendritic structures will be created numerically in d-dimensional Euclidian lattice space, and their size distribution and the fractal dimension (Mandelbrot:1982) will be considered. In the third section, we will discuss about the suitability of our model to the natural dendritic structures. A short comment on the relation to the problem of directed percolation (Grassberger: 1983) and diffusion limited aggregation (DLA) (Witten & Sander: 1983) which have been attracting much attention in the field of fractal theory will also be noted in that section.

§2 The model

First, we introduce a model of rivers so called Sheidegger's model (Sheidegger:1967). Let us assume the situation that rain is falling stationarily and uniformly on a slope. If the slope surface is not flat but has random irregularities, then fallen rain drops will walk down the slope with random fluctuations. When two rain drops collide, we regard them to be joined and make one drop. After the collision, the united drop is assumed to walk randomly just like before the collision. As a model of rivers, trajectories of each random walk and the collision points will be considered as tributary streams and confluences, respectively.

In order to analyse numerically, we discretize this model. Rains are assumed to fall on the nodes of the oblique lattice in Fig.1 and these points are regarded as sources of water of the same intensity on the plane. The ordinate x_1 is discretized to the slope and every flow drifts to that direction. In this discretized model, flows are also discretized on the bonds of the lattice. As a consequence of the stickiness of water drops, we prohibit any flow to split. Flows are allowed to go right down or left down only (see Fig.2). Here, the probability of each realization is 0.5.

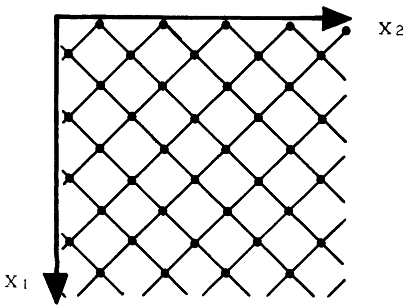

Fig. 1 The lattice. Water flows on the bonds.

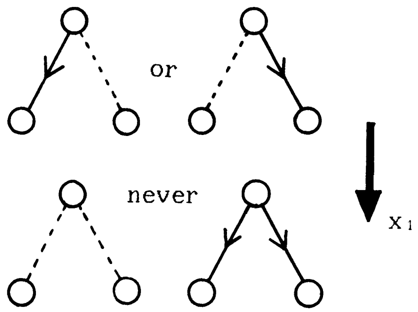

Fig. 2 Elementary flow patterns. Lower two patterns never ocuur.

Fig.3 shows an example of our simulation. We can find many dendritic structures distributing densely on the lattice. It may be obvious from the above generating rule that these rivers do not contain any loop and all branches are directed to the up stream (the direction of $-x_1$). Any two separating flows at upper stream are likely to join at lower stream, on the contrary, flows at lower stream never connect to another river at upper stream. Namely, we can distinguish each river at the lowest end.

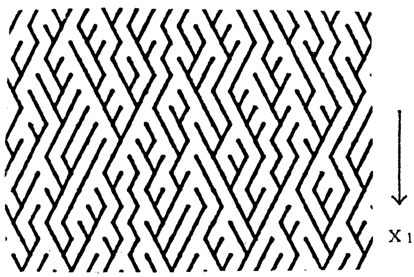

Fig. 3 An example of river pattern.

One of the most important quantity in this system is the size distribution of the rivers. Here, the size of a river is defined by the total number of nodes which belong to its upstream. Since every node is the source of water of the same intensity, s represents also the amount of flow of the river. If we denote the size of river at (m,n)-th node by s(m,n), then this quantity satisfies the following equation:

$$s(m+1,n) = r(m,n)\cdot(1+s(m,n)) + (1-r(m,n+1))\cdot(1+s(m,n+1)) \quad (1)$$

where r(m,n) is a random number which is equal to 1 when the flow at (m,n)-th node goes right down and is equal to 0 in the other case. Calculating Eq.1 by computer, we can obtain the distribution of s at sufficiently lower stream. Eq.1 is solved under periodic boundary condition on the x_1 axis, although the periodicity does not afect the following result if the system size in the x_2 direction is chosen to be larger than or equal to that in the x_1 direction. In Fig.4, the distribution of s is plotted as the case d=2. Here, the system size is about $10^4 \times 10^4$. It is very clear from this figure that the distribution of s satisfies the following power law in a wide range of s :

$$P(\geq s) \propto s^{-\alpha}, \quad \alpha = 0.331\pm0.006 \quad (2)$$

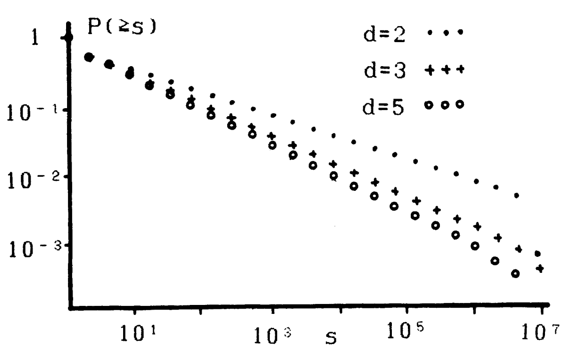

Fig. 4 The distribution of s. d denotes the spatial dimension.

Table 1

d	α	D
2	0.331±0.006	1.50
3	0.465±0.003	1.85
4	0.491±0.007	1.96
5	0.496±0.010	2.00
6	0.500±0.005	2.00
⋮	⋮	⋮
∞	0.499±0.009	2.00

where $P(\geq s)$ is the probability that the size of a randomly chosen river is larger than s. There is a higher cut-off in the distribution of s, above which the distribution of s decay more rapidly than the power law. This cut-off (denoted by s_c) is related to the system size in the x_1 direction (denoted by L) by the following relation:

$$s_c \propto L^{1/(1-\alpha)} \tag{3}$$

This can be deduced theoretically by considering the expectation value of s, <s>, in two ways: one is the value obtained by Eq.2 with a sharp cut-off at s_c, and the other is obtained by using the relation $\langle s \rangle \propto L$ which is required from the generating rule of s, Eq.1.

Since s_c is nearly equal to the size of the largest river, it is reasonable to define the fractal dimension of the river, D, from the scaling relation, Eq.3, as

$$D = \frac{1}{1-\alpha} \tag{4}$$

For smaller rivers, the fractal dimension is assumed to take the same value, because smaller rivers become parts of larger rivers if the system size is enlarged sufficiently. Substituting the numerical value $\alpha=0.331$ into Eq.4, we obtain the fractal dimension as $D \simeq 1.50$.

Spectrum of s in the x_2 direction is found to be white. This means that the spatial distribution of the rivers in our model is completely random.

Next, we extend this model to higher dimensional cases. The extension is natural and the model can be constructed in the same way.

Let us consider flows in d-dimensional Euclidian space $(x_1, x_2, \cdots, x_d)$. All points in the space are assumed to be sources of the flows of the same intensity. Just like the case of d=2, x_1 component of any flow vector is always positive while the other components take negative values with the probability 0.5. As a consequence, the flows may join but never split toward the x_1 direction, hence they make dendritic structures.

Discretized version of the above model for numerical calculation can be readily constructed and we can obtain the distribution of flow. It is confirmed that the distribution of s follows the power law in any case (see Fig.4 again). The exponent α in Eq.2 is found to depend on the spatial dimension d. In Tab.1, the values of α are listed with the fractal dimension. α increases with d while $d \leq 5$, however, there seems to be a critical dimension $d_c=5$ above which α takes a constant value 0.5. We can calculate the fractal dimension by using Eq.4, since Eq.4 is vallid independent of the spatial dimension d. It is very interesting that the fractal dimension of the dendritic flows never exeed 2 no matter how large the dimension of embeded space is. Note that, in this model, d-dimensional Euclidian space is densely packed with D-dimensional random dendritic fractals.

Theoretical explanation for the above results is not easy. We have succeeded only in the case of d=2 and $d=\infty$.

In the continuum limit, the flows in our model are constructed by Brownian motions drifting toward the x_1 direction. In the case d=2, the size of river, s, is proportional to the area of its drainage basin. And the drainage basin can be considered by ridges (see Fig.5). Here, the ridges are also Brownian motions, hence s_c is proportional to the area surrounded by two trajectories of Brownian motions. Roughly speaking, the area may be proportional to the product of the river's height and width. Regarding the x_1 axis as time axis, the distribution of the hight, L, can be obtained from the distribution of recurrence of Brownian motion :

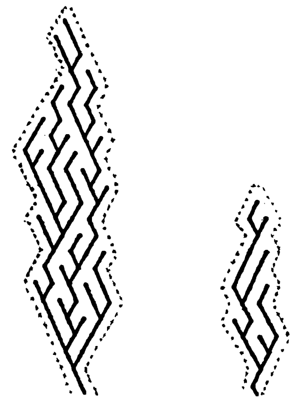

Fig. 5 River ———— and ridges ....

$$P(\geq L) = \int_L^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2m}} \cdot m^{-1/2}\, dm \quad . \qquad (5)$$

It is probable to assume that the width is nearly equal to $L^{1/2}$, hence, s is proportional to $L^{3/2}$. Substituting this scaling relation to Eq.5, we obtain the distribution of s as

$$P(\geq s) \propto s^{-1/3} \quad . \qquad (6)$$

Here, the exponent α is 1/3 which is very close to the numerical value α=0.331.

In the case d=∞ , we apply the mean field theory. It is convenient to regard the x_1 axis as time axis also, and we denote it by t in the following discussion. Time evolution of the distribution of s is governed by two elementary processes: one is the collision of flows and the other is the growth caused by the uniform rainfall. These factors can be described mathematically by the following partial differential equation for the probability density of s, p(s,t):

$$\frac{\partial}{\partial t}p(s,t)+c\frac{\partial}{\partial s}p(s,t) = (1-c)p(s,t)+ \frac{c-1}{2}\int_0^s p(s',t)p(s-s',t)ds' . \qquad (7)$$

Here, c is a constant and we have neglected higher order collisions. Stationary solution of Eq.7 can be solved analytically and p(s,·) is represented by a modified Bessel function. Then the cumulative distribution P(≥s) becomes

$$P(\geq s) \propto s^{-1/2} \quad . \qquad (8)$$

This is consistent with the numerical result in the case d= ∞ .

§3 Discussions

As mentioned in the previous sections, our model in the case d=2 is introduced as a model of rivers. The fractal dimension of rivers predicted by our model is 1.5, while that of natural rivers is not universal but distributing from 1.4 to 1.8 (Kunigami:1985). This means that our model is not bad, however, it is too much simplified to explain the diversity of rivers in nature. We should include some geometrical factors such as the roughness of earth's releaf in order to improve our model.

In the three dimensional case, we hope our model resembles blood vessels. The diameter distribution of blood vessels is known to obey a power law. In Fig.6, the cumulative distribution of blood vessels are plotted with the diameter, r, on logarithmic graph paper using Wiedeman's data (Wiedeman:1963). We obtain the following distribution:

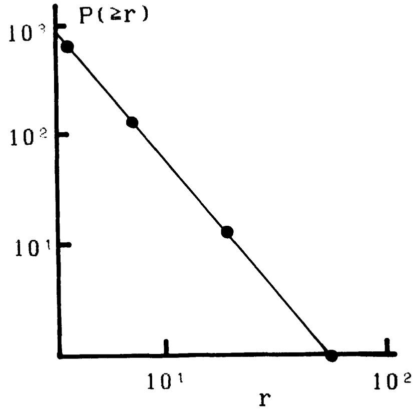

Fig. 6 Diameter distribution of blood vessels.

$$P(\geq r) \propto r^{-\beta} \quad , \qquad \beta \simeq 2.3 \qquad (9)$$

In order to relate this power law to our model, we have to find the relation between the diameter and blood flow quantity, s. When a viscous fluid flows through a pipe having a diameter r, the amount of flow is solved analyticaly by Hagen-Poiseuille theory as

$$s = \frac{2\pi p'}{\mu} \cdot r^4 \quad , \qquad (10)$$

where μ and p' denote the viscosity and the pressure gradient, respectively. The pressure gradient p' is likely to be reduced for thinner blood vessels. If we assume that p' is proportional to r, then s becomes proportional to r^5. This relation together with Eq.9 leads the following distribution of s:

$$P(\geq s) \propto s^{-\beta/5} \quad , \qquad \beta/5 \simeq 0.46 \quad . \qquad (11)$$

It is surprising that the exponent 0.46 coincides excellently with α in our model in the case d=3. Although this coincidence depends on the above assumption of p', we may say that our model is good for blood vessels. This indicates that dendritic structures of blood vessels are created by simple random aggregation of conservative flows.

Another candidate for our model may be the tree. The fractal dimension of trees is known to range from 1.3 to 1.8 (Morse et al.:1985), however, that of our model is 1.85 in the case d=3. Our model is not good for trees on the point that it creates more complicated dendritic structures than actual trees.

Lastly, we discuss about conections with directed percolation and DLA.

In directed percolation, all four patterns in Fig.2 occures stochastically. The probability of each realization is p(1-p), (1-p), $(1-p)^2$ and p^2 for left up case, right up case, left down case and right down case, respectively, where p is a controle parameter ranging from 0 to 1. The probabilities of the latter two cases can not vanish simultaneously, hence, directed percolation is different from our model for any value of p. It should be noted here that the prohibition of the latter two cases , i.e. extinction and division of flows, are neccesary for the power law in Eq.2. If we allow the extinction or division with a certain probability, then the power law will be replaced by the exponential distribution.

DLA considers aggregation of particles, on the other hand, our model is based on the aggregation of flows. However, if we regard the drifting axis (x_1) as time axis, then our model becomes a model of random aggregation of particles as mentioned in §2. From this standpoint, DLA and our model have deep connection to each other. The interest of DLA is mainly focused on geometrical structure of aggregated particles, contrary to this, that of our model is focused on the distribution of the particles. Note that in our model scattering cross sections are constant and independent of size of particles. Namely, we neglect some factors such as geometrical effects and mass contribution in our model.

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1-2

Q: What is the relationship of your model to directed percolation ? (L. Sander)

A: In an ordinary percolation problem, fractal structures are obtained only at the critical phase transition point. Contrary to this, in our model there is no phase transition and we can always find at least one percolation cluster among fractal clusters. The percolation cluster is completely directed, namely all branches are directed to the upper stream. In this sense our model is related to the directed percolation.

Q: In your theory, the existence of conservative flows is important. Then, what are conservative flows in case of trees ? (K. Kitahara)

A: The flows of sap in branches may be conserved, however, as I mentioned, our model is not so good for trees.